Suppose we decided to count things by packing them in cartons of 5. If we had 89 tomatoes, we would make 17 cartons of 5, with 4 loose tomatoes. Now suppose that we decided to further organize things by packing every 5 cartons into a tray. How many trays would we use? We would use 3 trays and have 2 loose cartons, and the 4 loose tomatoes.

How many tomatoes are in each tray? There are 5 cartons of 5 tomatoes, or 25 tomatoes, in each tray. Now suppose we decided to go even further and pack every 5 trays into a crate. How many tomatoes would each crate hold? Each would hold 5 trays of 5 boxes of 5 tomatoes, or $5^3$. In other words, each crate holds 125 tomatoes.

What we have described is a base 5 number system. Our discussion shows that eighty-nine represented in base 5 is written as 324. To avoid confusion, we will write

$$89 \text{ (dec)} = 324 \text{ (base 5)}$$

Notice that each position has a different meaning associated with it. We consider each place to have a value, so we call this a place value system. The key to this system is the number 5, called the base. Knowing this, we can calculate as many place values as we need. The lowest place value (in any base system) is 1. Each place value is equal to the previous place value times the base. In base 5, the place values are 1, 5, 25, 125, 625, 3125, and so on.

Remember, 324 (base 5) is eighty-nine, not three hundred twenty-four. For instance, the “3” stands for 3 twenty-fives, not 3 hundreds. It may help us clarify the situation by saying that the “3” has a face value of three and a place value of twenty-five, for a value of seventy-five.

Example 1: Convert 1423 (base 5) to decimal.

Remember base 5 place values: 1, 5, 25, 125,…

$$1423 \text{ (base 5)} = 1(125) + 4(25) + 2(5) + 3(1)$$
$$= 125 + 100 + 10 + 3$$
$$= 238 \text{ (dec)}$$

Example 2: Express 266 (dec) in base 5.

Remember base 5 place values: 1, 5, 25, 125, 625, etc.
Think—we don’t need any 625’s (or any bigger place values) to make 266.

How many 125’s do we need? $266 / 125 = 2$, leaving a remainder of 16.
How many 25’s do we need? $16 / 25 = 0$, still leaving a remainder of 16.
How many 5’s do we need? $16 / 5 = 3$, leaving a remainder of 1.
How many 1’s do we need? 1.

Conclusion: $266 \text{ (dec)} = 2031 \text{ (base 5)}$
We could have packed things with any number in each bundle, so any integer larger than 1 can be a base.
Let’s try base 7. Think of packing with 7 in a carton, 7 cartons in a tray, and so on.

Example 3:
Convert 5026 (base 7) to decimal.

base 7 place values: 1, 7, 49, 343

5026 (base 7) = 5(343) + 0(49) + 2(7) + 6(1)  
= 1715 + 0 + 14 + 6  
= 1735 (dec)

Example 4:
Express 987 in base 7.

Remember base 7 place values: 1, 7, 49, 343, 2401, etc.
Think--we don’t need any 2401’s (or any bigger place values) to make 987.

How many 343’s do we need? 987 / 343 = 2, leaving a remainder of 301.
How many 49’s do we need? 301 / 49 = 6, leaving a remainder of 7.
How many 7’s do we need? 7 / 7 = 1, leaving a remainder of 0.
How many 1’s do we need? 0.

Conclusion: 987 (dec) = 2610 (base 7)

Remarks:
Notice that in base 5, we only need 5 different symbols: 0, 1, 2, 3, and 4.
When counting in base 5, the biggest number that fits in one place is 4.
The smallest number that needs two places is 10 (base 5), which means five.
The biggest number that fits in two places is 44 (base 5), which means twenty-four.
The smallest number that needs three places is 100 (base 5), which means twenty-five.

Notice that in base 7, we only need 7 different symbols: 0, 1, 2, 3, 4, 5, 6.
When counting in base 7, the biggest number that fits in one place is 6.
The smallest number that needs two places is 10 (base 7), which means seven.
The biggest number that fits in two places is 66 (base 7), which means forty-eight.
The smallest number that needs three places is 100 (base 7), which means forty-nine.