# Chapter 9

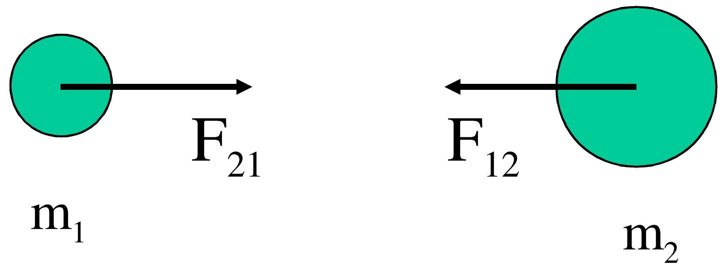
## Linear Momentum

# Linear Momentum

- Conservation of Linear Momentum
- Kinetic Energy of a System
- Collisions
- Collisions in Center of Mass Reference Frame

# Momentum From Newton's Laws

Consider two interacting bodies with $m_2 > m_1$:




If we know the net force on each body then $$\Delta \mathbf{v} = \mathbf{a}\Delta t = \frac{\mathbf{F}_{\text{net}}}{m}\Delta t$$

The velocity change for each mass will be different if the masses are different.

# Momentum

Rewrite the previous result for each body as:

$$m_1 \Delta \mathbf{v}_1 = \mathbf{F}_{21} \Delta t$$

$$m_2 \Delta \mathbf{v}_2 = \mathbf{F}_{12} \Delta t = -\mathbf{F}_{21} \Delta t$$ From the 3rd Law

Combine the two results:

$$m_1 \Delta \mathbf{v}_1 = -m_2 \Delta \mathbf{v}_2$$

$$m_1 \left( \mathbf{v}_{1f} - \mathbf{v}_{1i} \right) = -m_2 \left( \mathbf{v}_{2f} - \mathbf{v}_{2i} \right)$$

# Momentum

Simplified notation:

$$Before: v_1, v_2$$
$$After: u_1, u_2$$

Rewrite the previous results with new notation:

$$m_1\left(\mathbf{v}_{1f} - \mathbf{v}_{1i}\right) = -m_2\left(\mathbf{v}_{2f} - \mathbf{v}_{2i}\right)$$

$$m_1\left(u_1 - \mathbf{v}_1\right) = -m_2\left(u_2 - \mathbf{v}_2\right)$$

# Define Momentum

The quantity (m**v**) is called momentum (**p**).

**p** = m**v** and is a vector.

The unit of momentum is kg m/s; there is no derived unit for momentum.

# Momentum

From a previous slide:

$$m_1\Delta\mathbf{v}_1 = -m_2\Delta\mathbf{v}_2$$

$$\Delta\mathbf{p}_1 = -\Delta\mathbf{p}_2$$

The change in momentum of the two bodies is “equal and opposite”. **Total momentum is conserved during the interaction; the momentum lost by one body is gained by the other.**

$$\Delta\mathbf{p}_1 + \Delta\mathbf{p}_2 = \mathbf{0}$$

# Momentum With Calculus

$$\vec{p} = m\vec{v}$$

$$\frac{d\vec{p}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

$$\therefore \vec{F}_{net} = \frac{d\vec{p}}{dt}$$

The above is not true, in this form, if the mass is changing, i.e. a rocket using up its fuel.

# Center of Mass Momentum

$$\vec{P}_{sys} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = M\vec{V}_{cm}$$

$$\frac{d\vec{P}_{sys}}{dt} = M\,\frac{d\vec{V}_{cm}}{dt} = M\vec{A}_{cm}$$

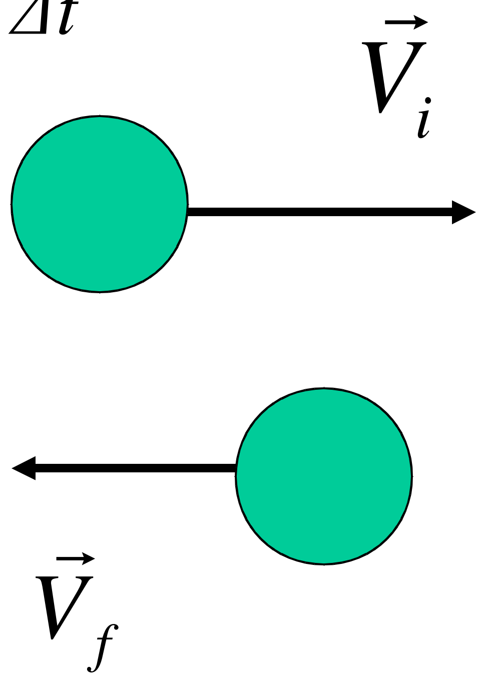
$$\sum_i \vec{F}_{ext} = \vec{F}_{net\ ext} = \frac{d\vec{P}_{sys}}{dt}$$

$$If\ \ \sum_i \vec{F}_{ext} = 0\ \ then$$

$$\vec{P}_{sys} = \sum_i m_i \vec{v}_i = M\vec{V}_{cm} = Constant$$

# Impulse

$$\Delta\vec{P} = \vec{P}_f - \vec{P}_i = M\left(\vec{V}_f - \vec{V}_i\right) = \vec{F}_{Avg}\Delta t$$




$$V_i = 20\,m/s;\ \ V_f = -15\,m/s;\ \ M = 3kg$$

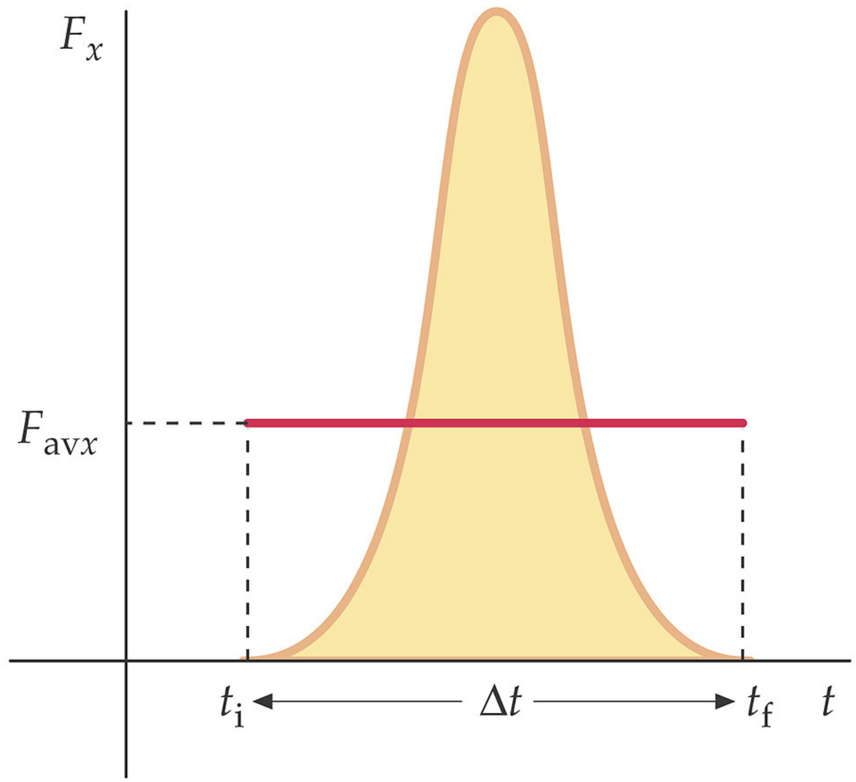
$$\Delta P = M\left(V_f - V_i\right) = 3(-15 - 20)$$

$$\Delta P = 3\left(-35\right) = -105\,kg \bullet m/s$$

Since we are applying the change of momentum over a time interval consisting of the instant before the impact to the instant after the impact, we ignore the effect of gravity over that time period

# Impulse and Momentum

The force in impulse problems changes rapidly so we usually deal with the average force



# Work and Impulse Comparison

**Work changes $E_T$**

Work = 0 ⟶ $E_T$ is conserved

Work = Force x Distance

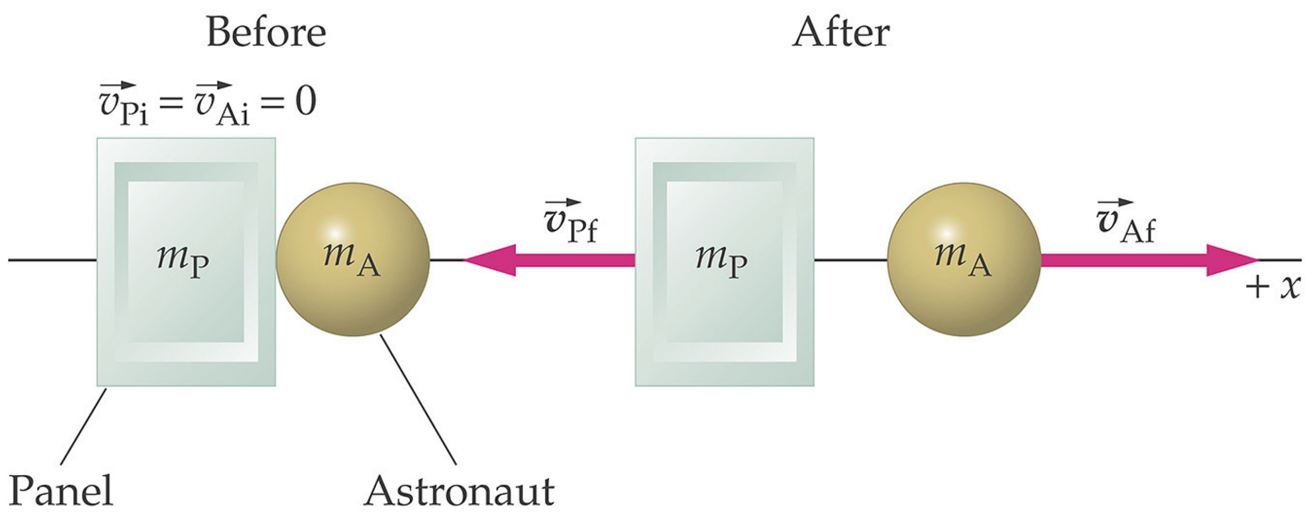
**Impulse changes** $\vec{P}$

Impulse = 0 ⟶ $\vec{P}$ is conserved

Impulse = Force x Time

# Typical Momentum Problems

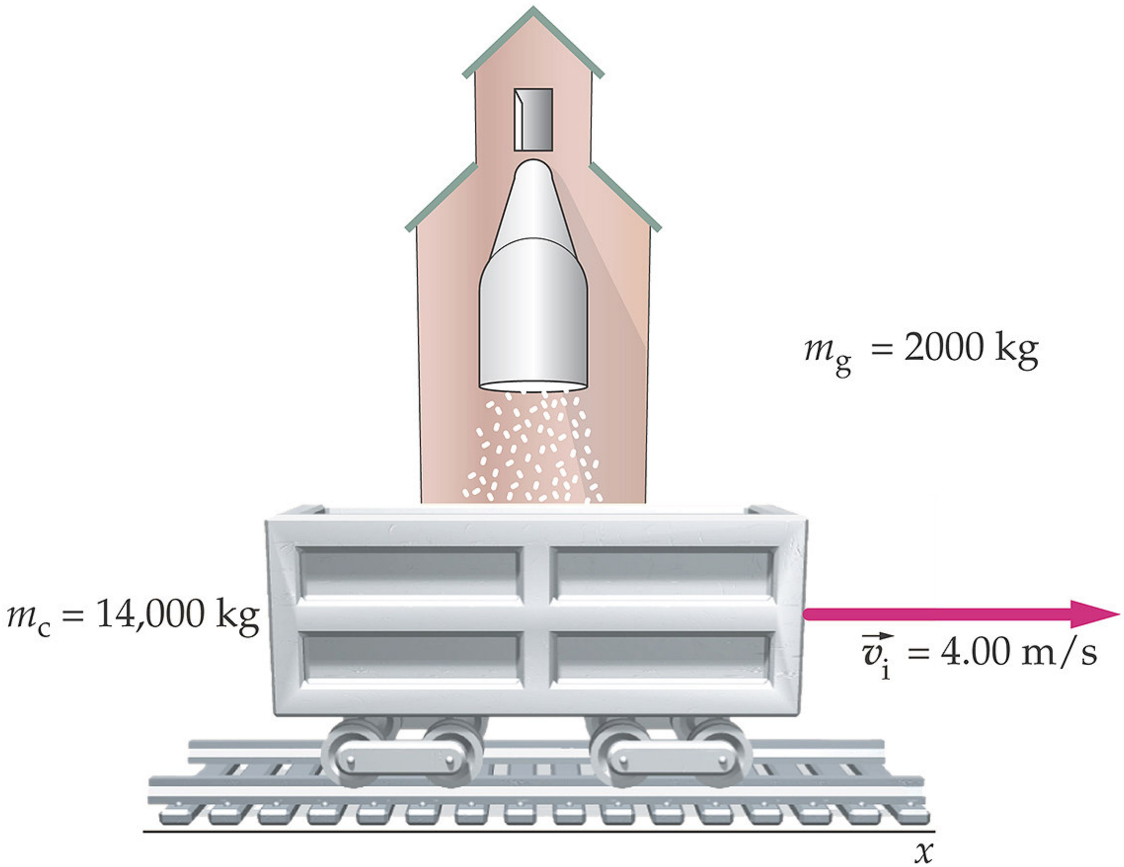
# Generic Zero Initial Momentum Problem



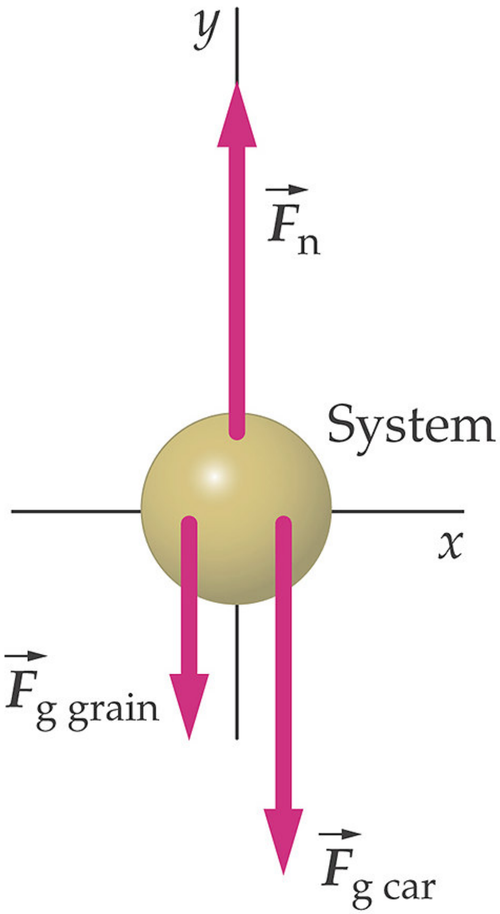

$$\sum \vec{p}_i = \vec{p}_P + \vec{p}_A = 0 \qquad \sum \vec{p}'_i = \vec{p}'_P + \vec{p}'_A = 0$$

These problems occur whenever all the objects in the system are initially at rest and then separate.

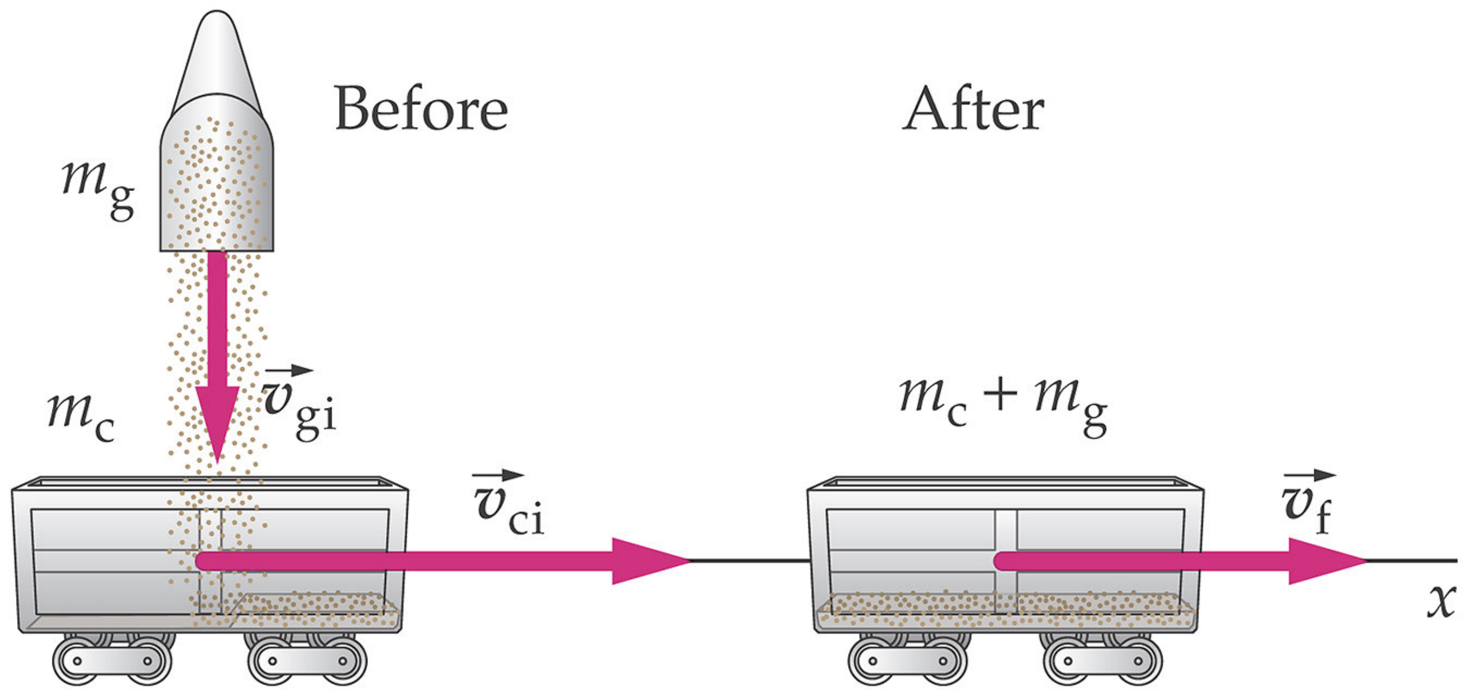
# Changing the Mass While Conserving the Momentum



# Changing the Mass While Conserving the Momentum

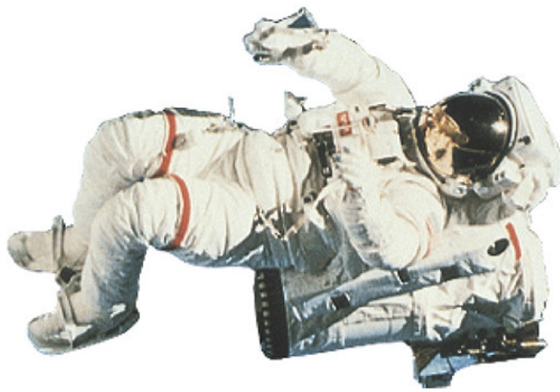


# Changing the Mass While Conserving the Momentum



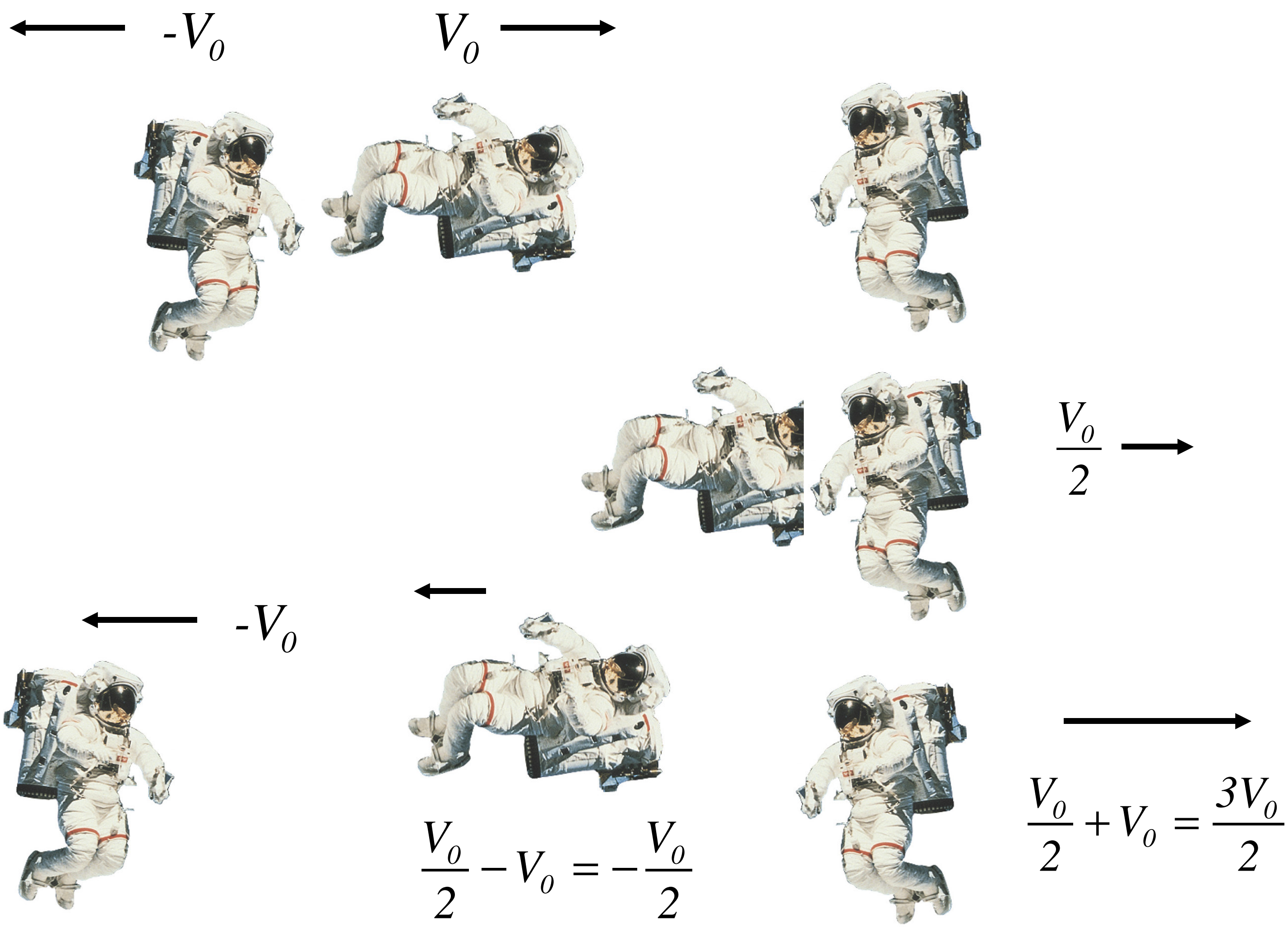
# Astronaut Tossing

## How Long Will the Game Last?

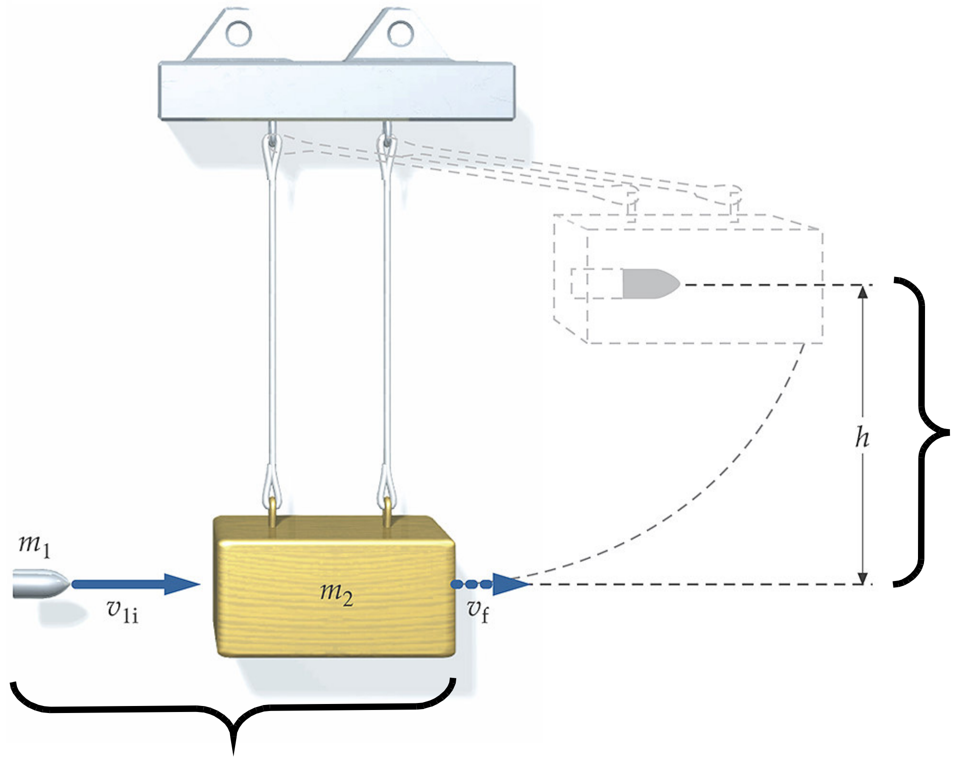


Assume all astronauts are the same size and mass. Also assume all astronauts have the same strength and push with equal speed.

# How Long Will the Game Last?



# The Ballistic Pendulum

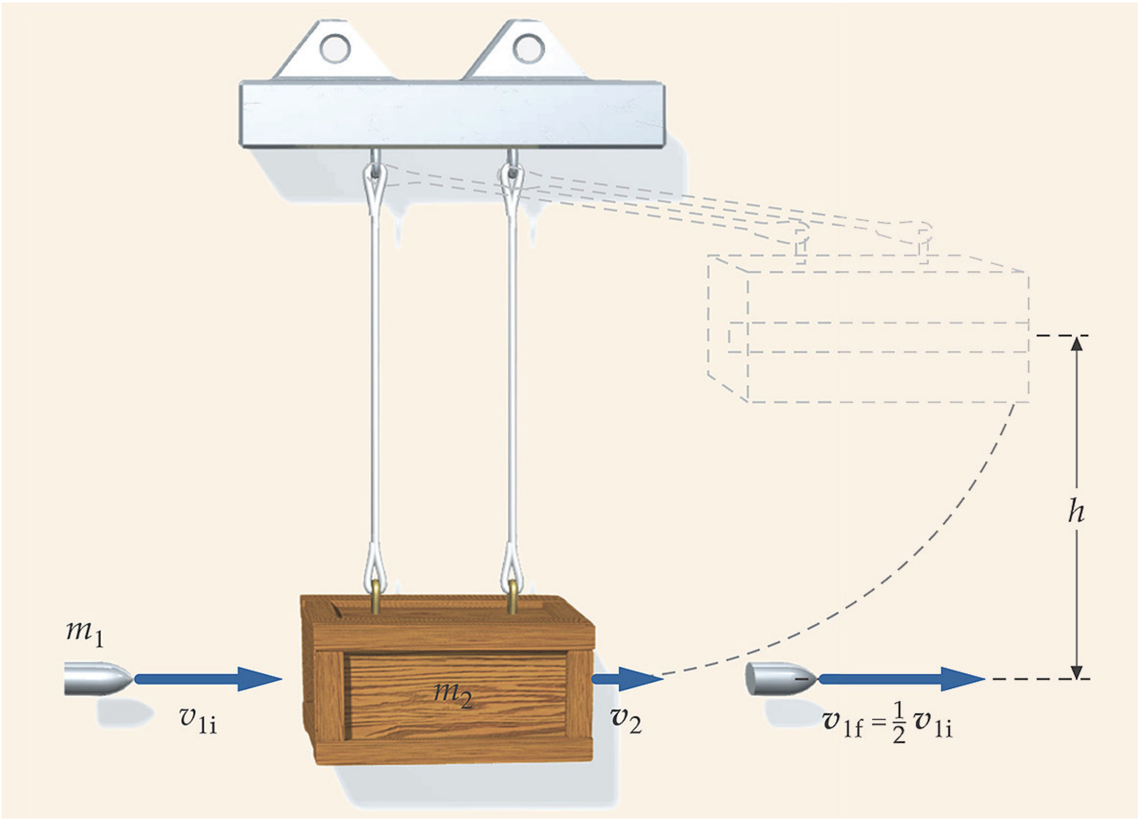



Conservation of Energy

Conservation of Momentum

Bullet remains in the block

# The High Velocity Ballistic Pendulum




Bullet exits the block

# Relative Speed Relationship

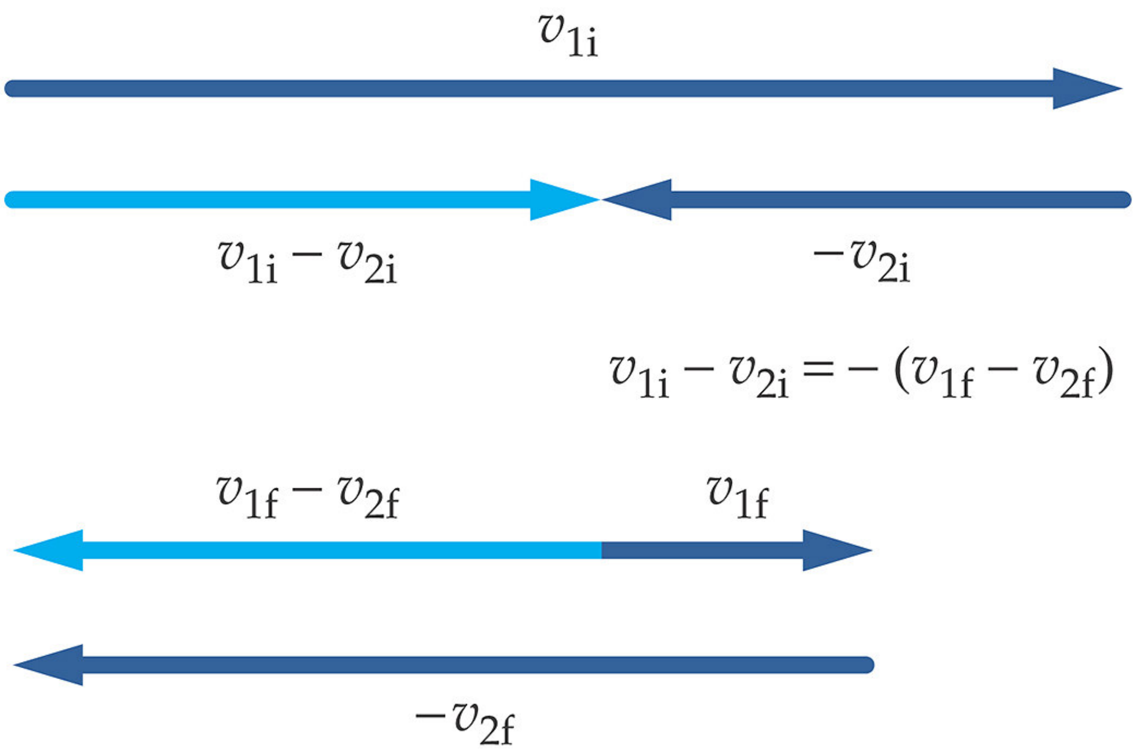
The relative speed relationship applies to *elastic collisions*.

In elastic collisions the total kinetic energy is also conserved.

If a second equation is needed to solve a momentum problem, the relative speed relationship is easier to deal with than the KE equation.

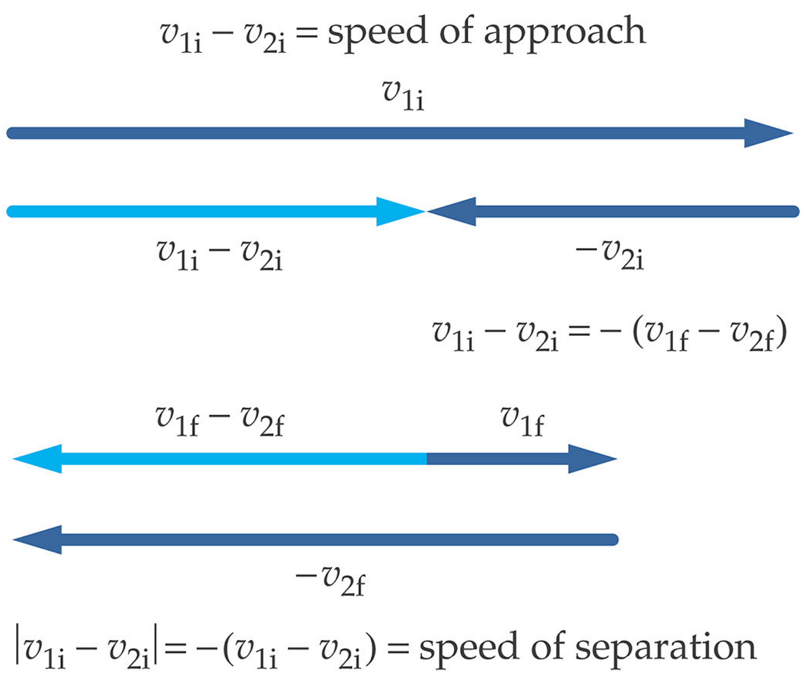
# Relative Speed Relationship

$$v_{1i} - v_{2i} = \text{speed of approach}$$




$$|v_{1i} - v_{2i}| = -(v_{1i} - v_{2i}) = \text{speed of separation}$$

# Relative Speed in Problem Solving

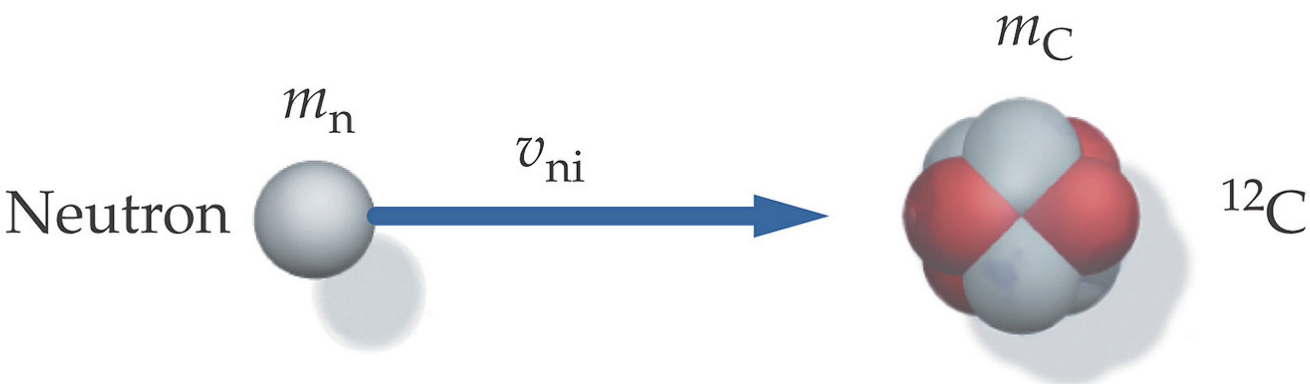



Need a second equation when solving for two variables in an elastic collision problem?

The relative speed relationship is preferred over the conservation of kinetic energy relationship

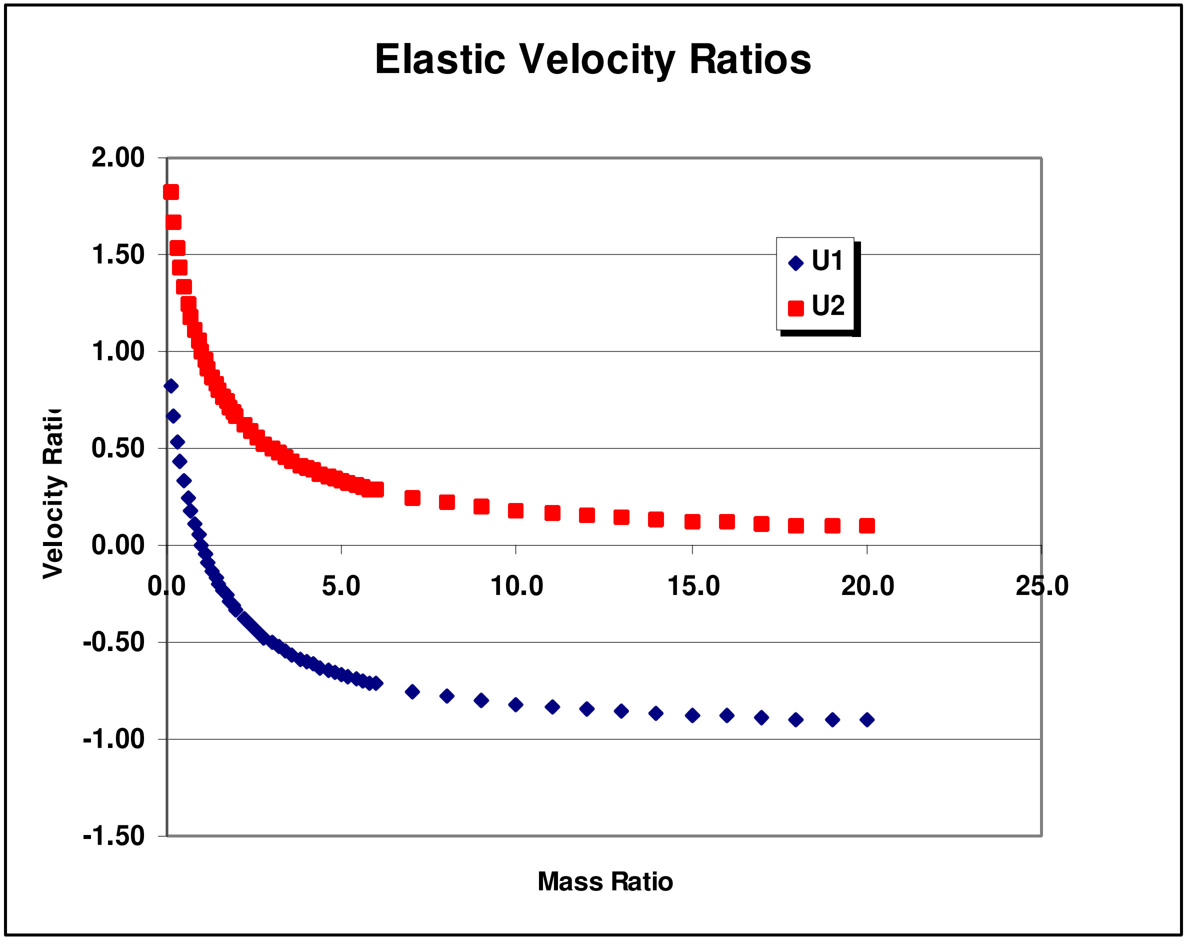
In the simpler notation: $$v_1 - v_2 = -(u_1 - u_2)$$

# Elastic Collision - Unequal Masses




In the later slides, the initially moving mass, $m_n$ , will called be $m_1$ while the initially stationary mass, $m_c$ , will be $m_2$.

# Elastic Collision – Velocity Ratios

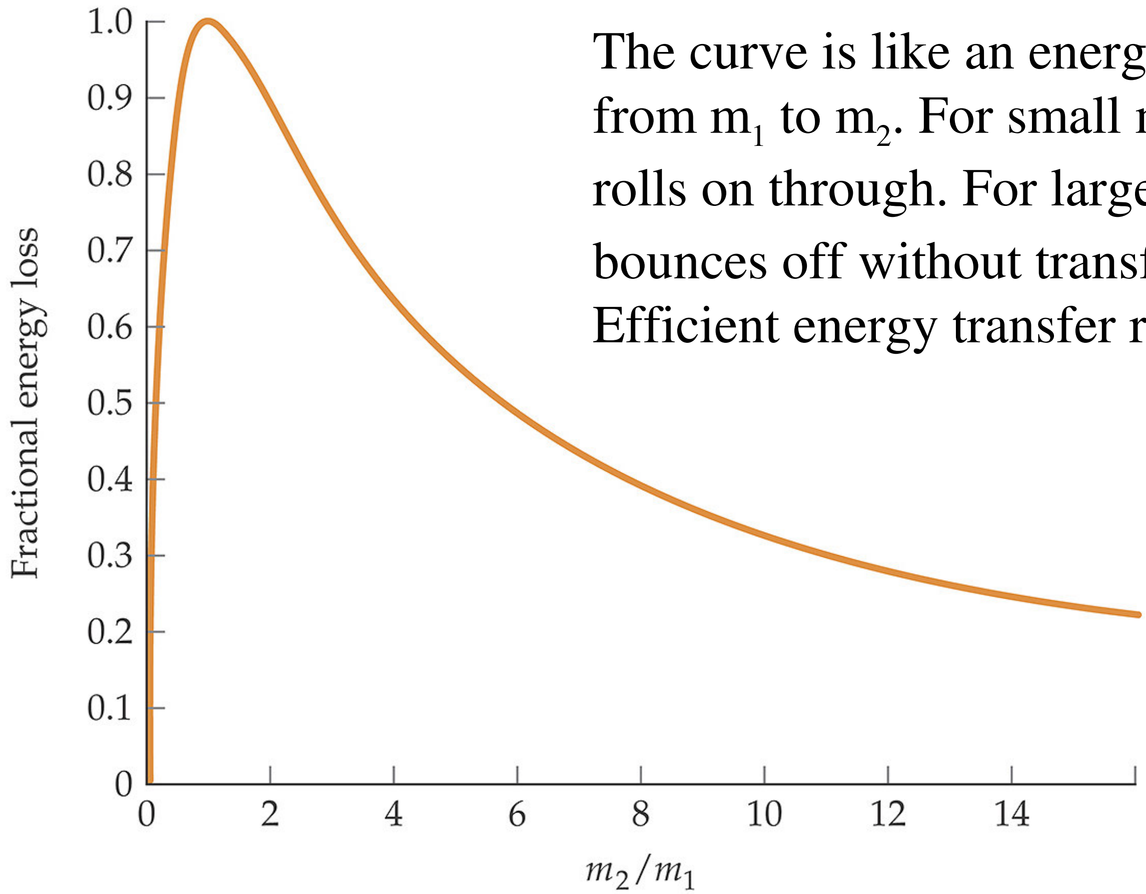



$$\frac{u_1}{v_0} = -\frac{\frac{m_2}{m_1} - 1}{1 + \frac{m_2}{m_1}}$$

$$\frac{u_2}{v_0} = \frac{2}{1 + \frac{m_2}{m_1}}$$

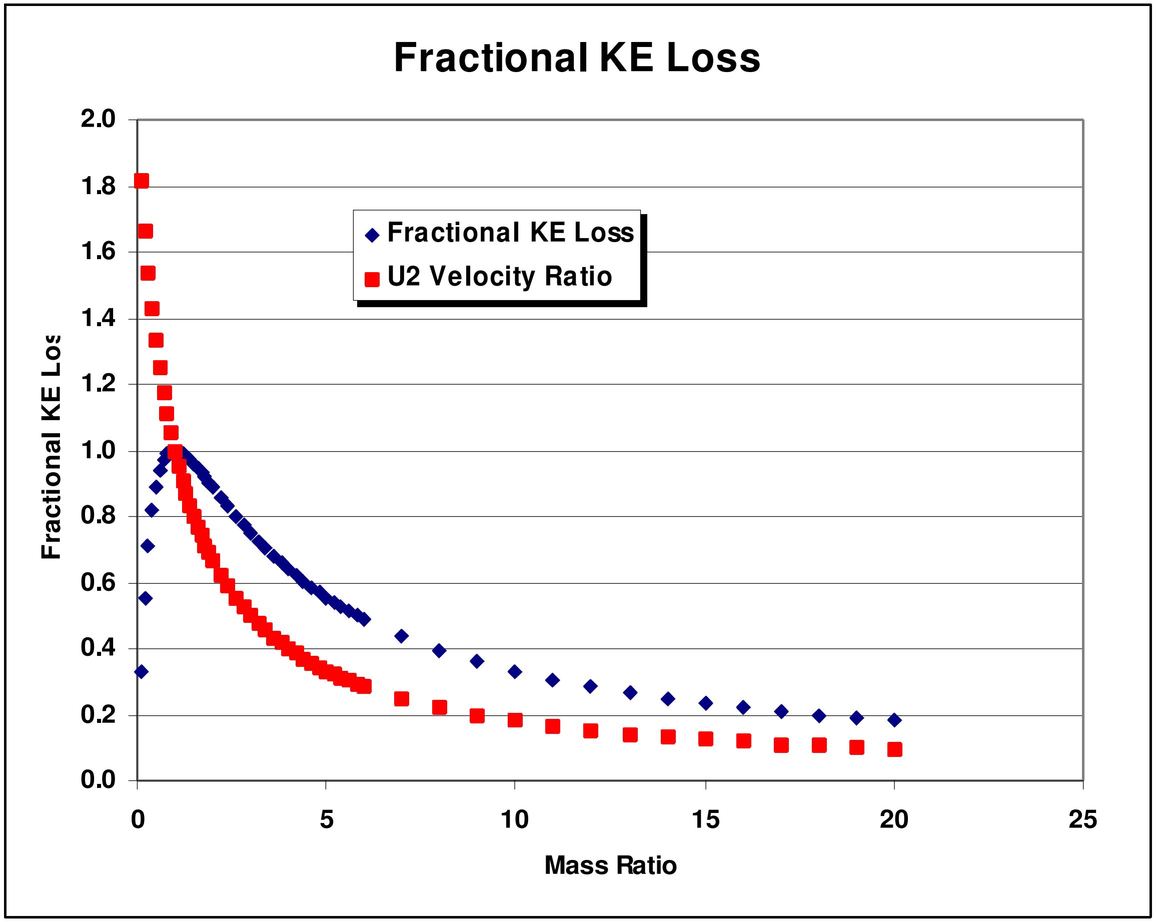
# Elastic Collision - Fractional KE "Loss"

For equal masses all of the KE is transferred from $m_1$ to $m_2$




The curve is like an energy transfer function from $m_1$ to $m_2$. For small $m_2$ the mass $m_1$ just rolls on through. For large $m_2$ the smaller $m_1$ just bounces off without transferring much energy. Efficient energy transfer requires equal masses.

# Elastic Collision - KE Transfer

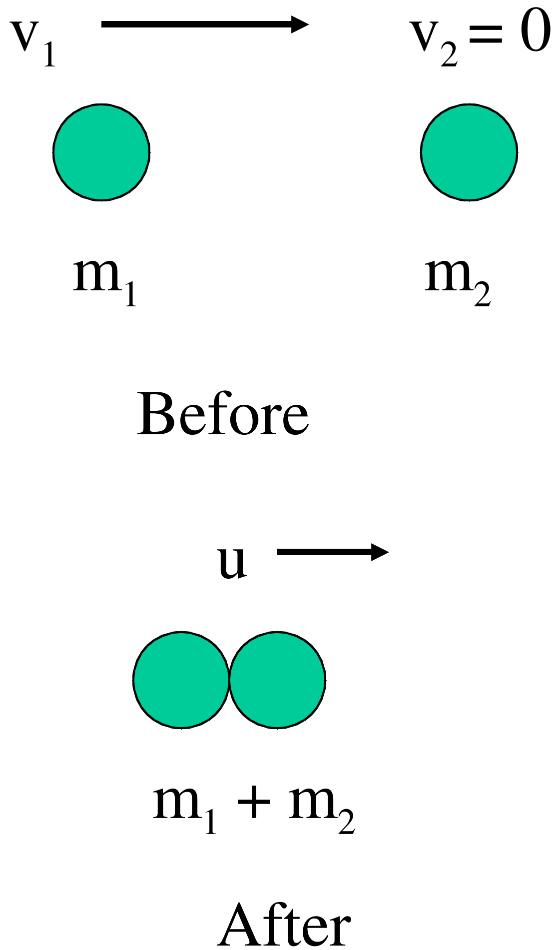



$$\frac{u_2}{v_0} = \frac{2}{1 + \frac{m_2}{m_1}}$$

$$f = \frac{4\frac{m_2}{m_1}}{\left(1 + \frac{m_2}{m_1}\right)^2}$$

For $m_2 > m_1$ the velocity of the 2nd mass gets a smaller velocity and even the larger mass can't overcome the shrinking $v^2$ factor in $KE_2$

# Inelastic Collision - Equal Masses




Momentum

$$m_1 v_1 = (m_1 + m_2)u; \quad u = \frac{m_1}{m_1 + m_2} v_1$$

Kinetic Energy

$$KE_i = \tfrac{1}{2} m_1 v_1^2; \quad KE_f = \tfrac{1}{2}(m_1 + m_2)u^2$$

$$KE_f = \tfrac{1}{2}(m_1 + m_2)\left[\frac{m_1}{m_1 + m_2}\right]^2 v_1^2$$

$$KE_f = \tfrac{1}{2}\frac{m_1}{m_1 + m_2} m_1 v_1^2 = \frac{m_1}{m_1 + m_2} KE_i$$

$$\frac{KE_f}{KE_i} = \frac{m_1}{m_1 + m_2}$$

# Elastic and Inelastic Collisions

Elastic Collision

$\vec{P}_{Total}$ is conserved

Total KE is conserved

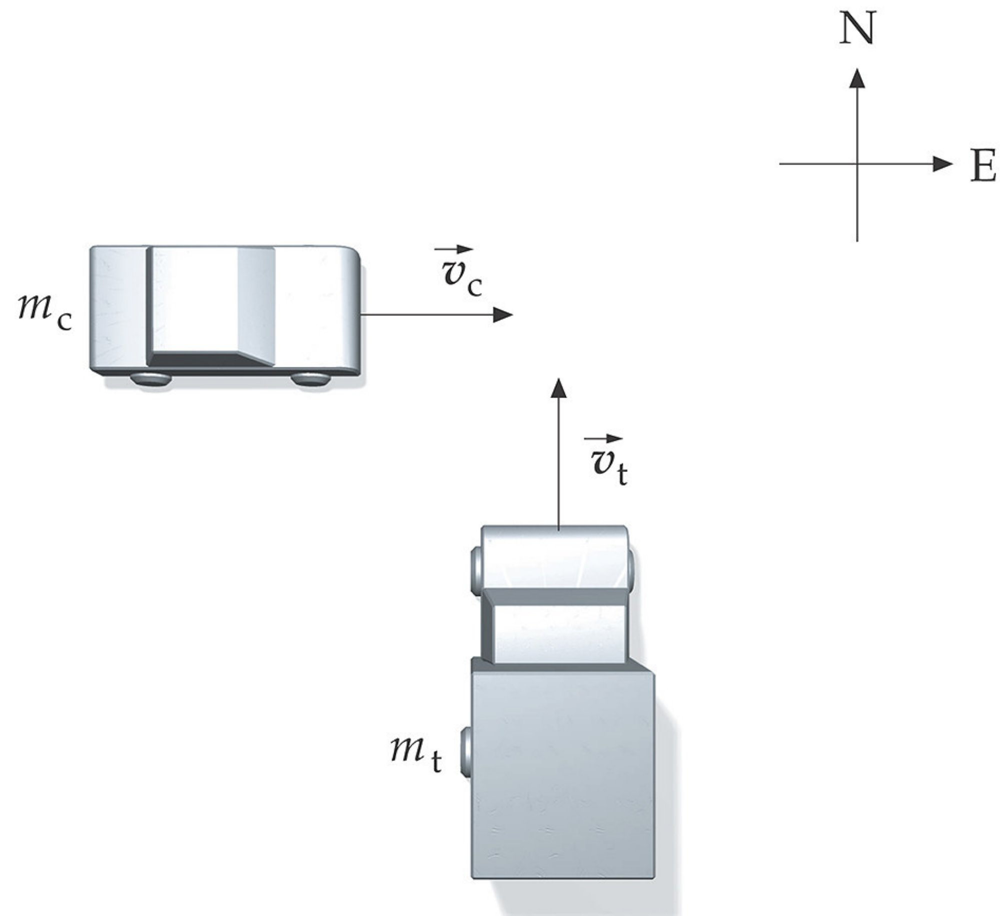
Inelastic Collision

$\vec{P}_{Total}$ is conserved

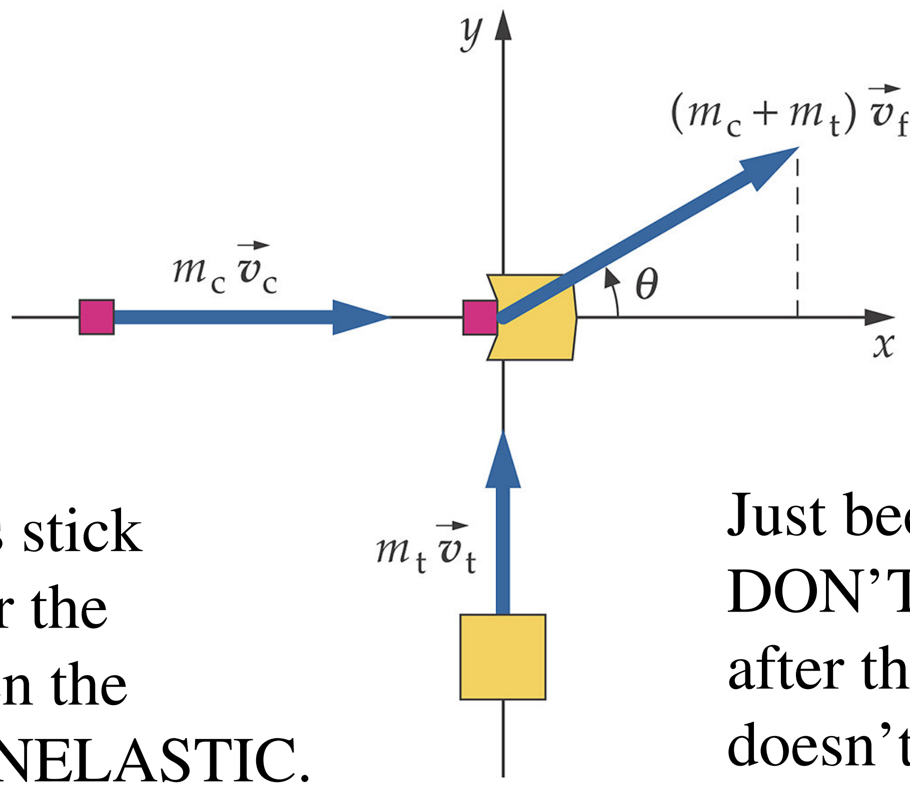
Total KE is NOT conserved

# Collisions in Two and Three Dimensions

# Totally Inelastic Collision



# Totally Inelastic Collision

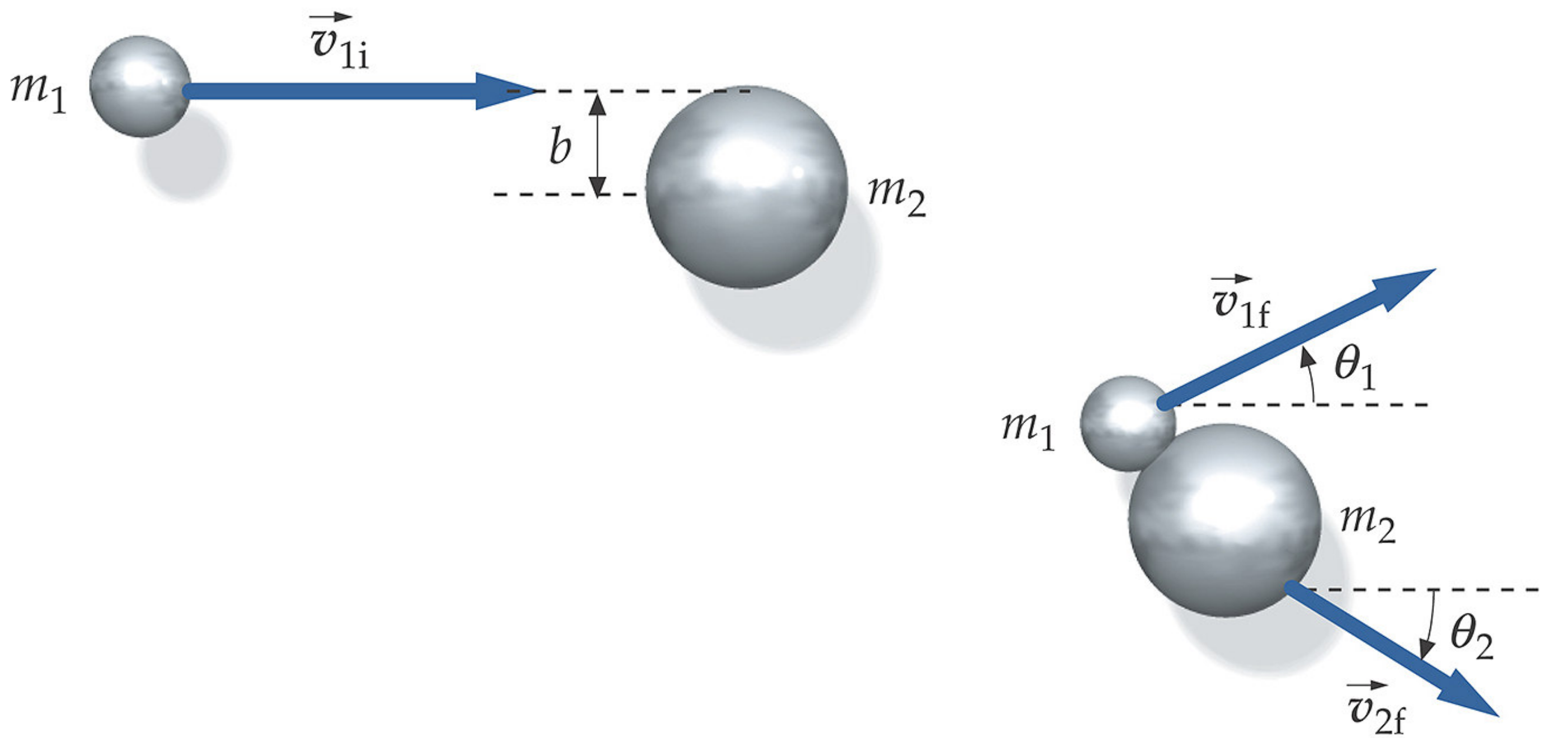



If the objects stick together after the collision, then the collision is INELASTIC.

Just because the objects DON'T stick together after the collision, doesn't mean the collision is ELASTIC.

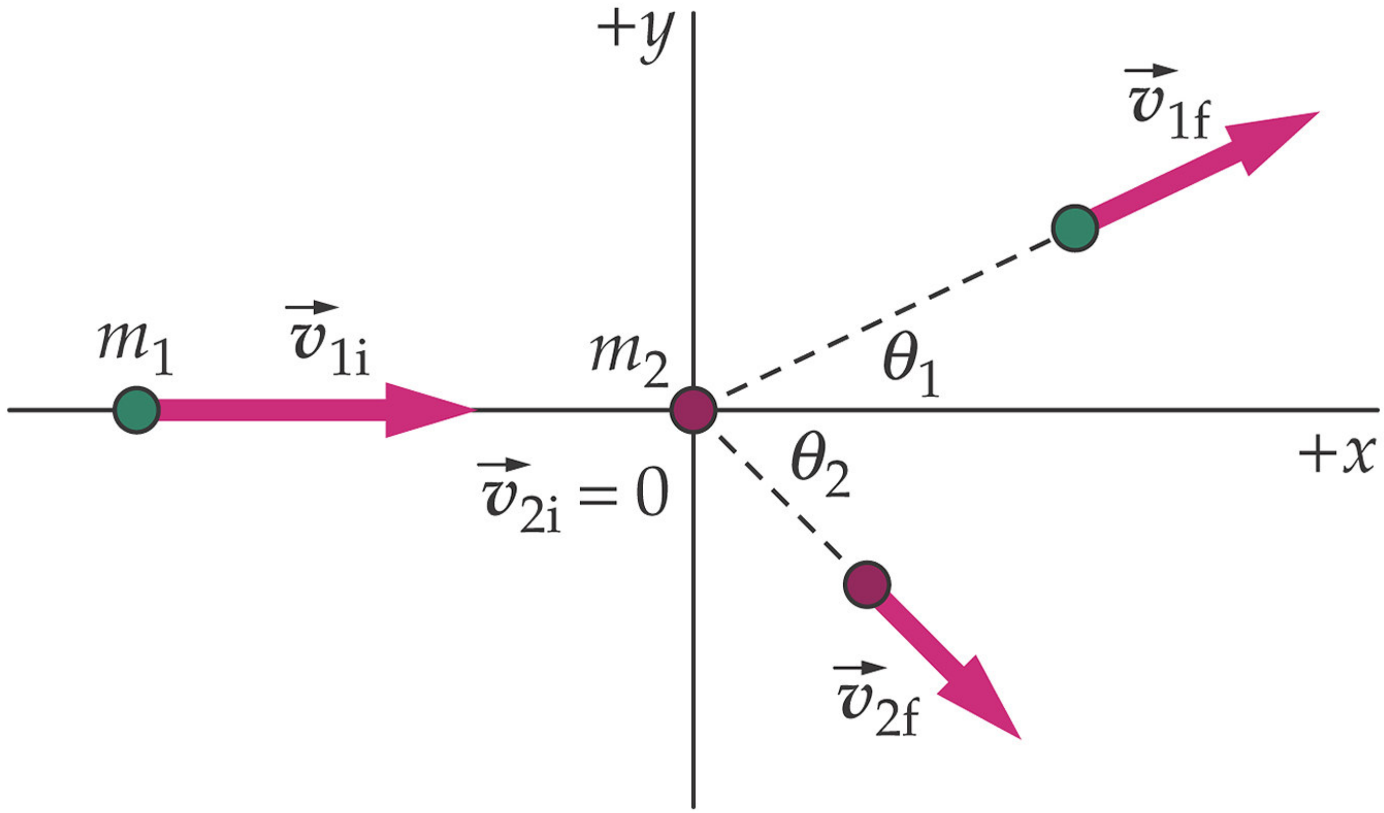
# General Glancing Collision

Initially only one object is moving



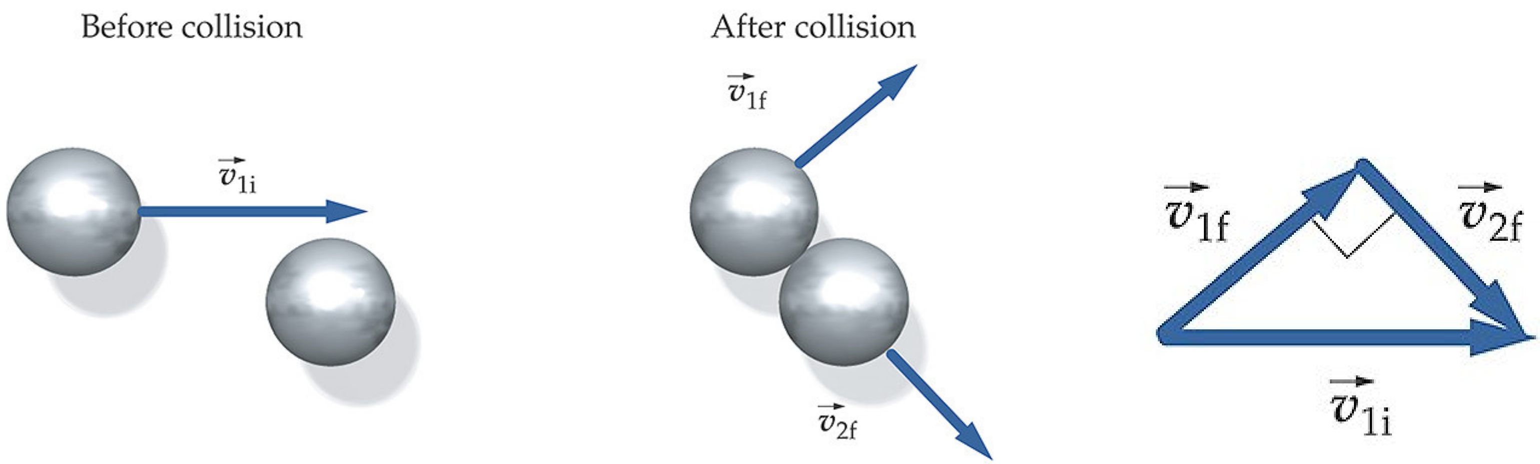
# General Glancing Collision

Initially only one object is moving



# Another 3-Vector Problem

## Special Case – Equal Masses



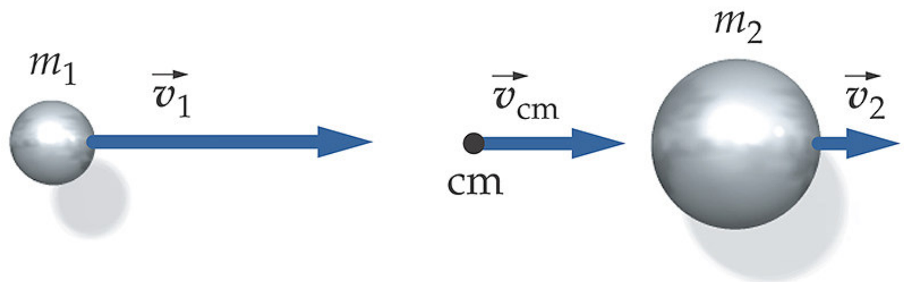

(a) (b)

Momentum problems where one of the objects is initially at rest yield 3-vector problems that can be solved using triangles, instead of simultaneous equations.

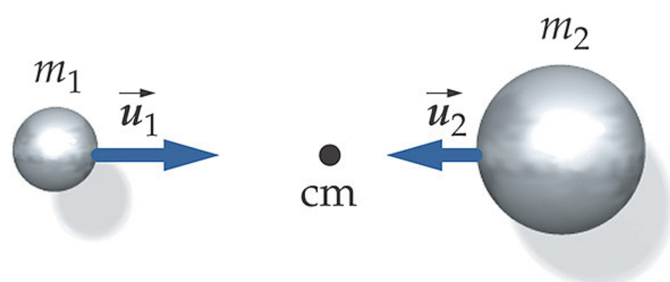
# Center of Mass Reference System in Momentum Problems

# Center of Mass Reference System

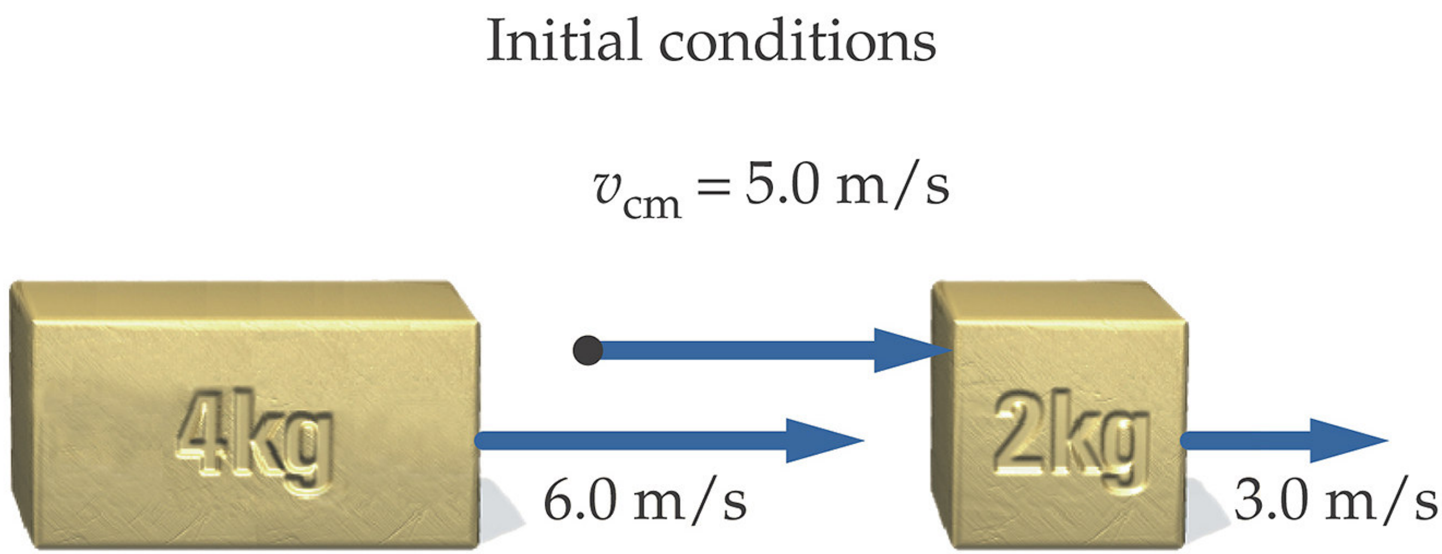
Original reference frame



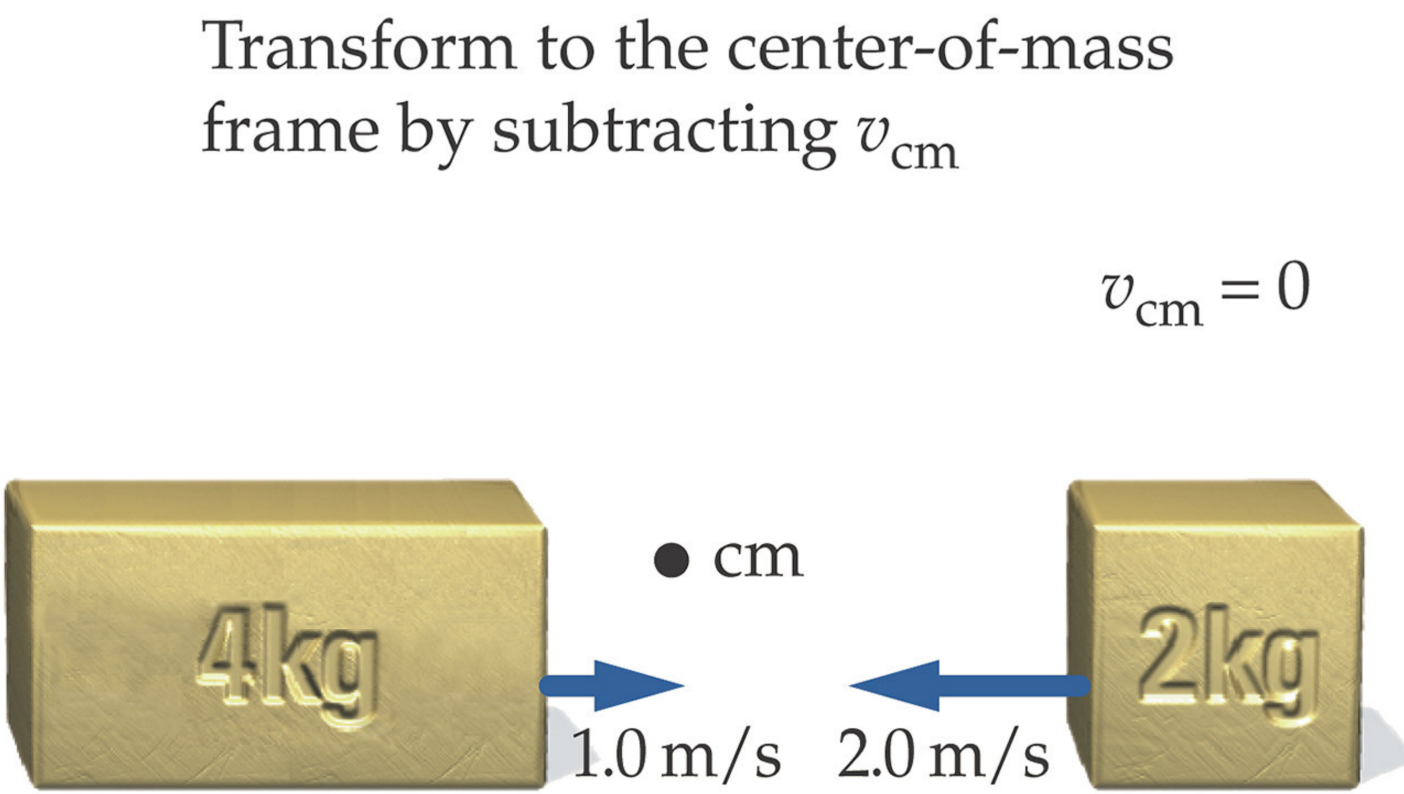

Center-of-mass reference frame



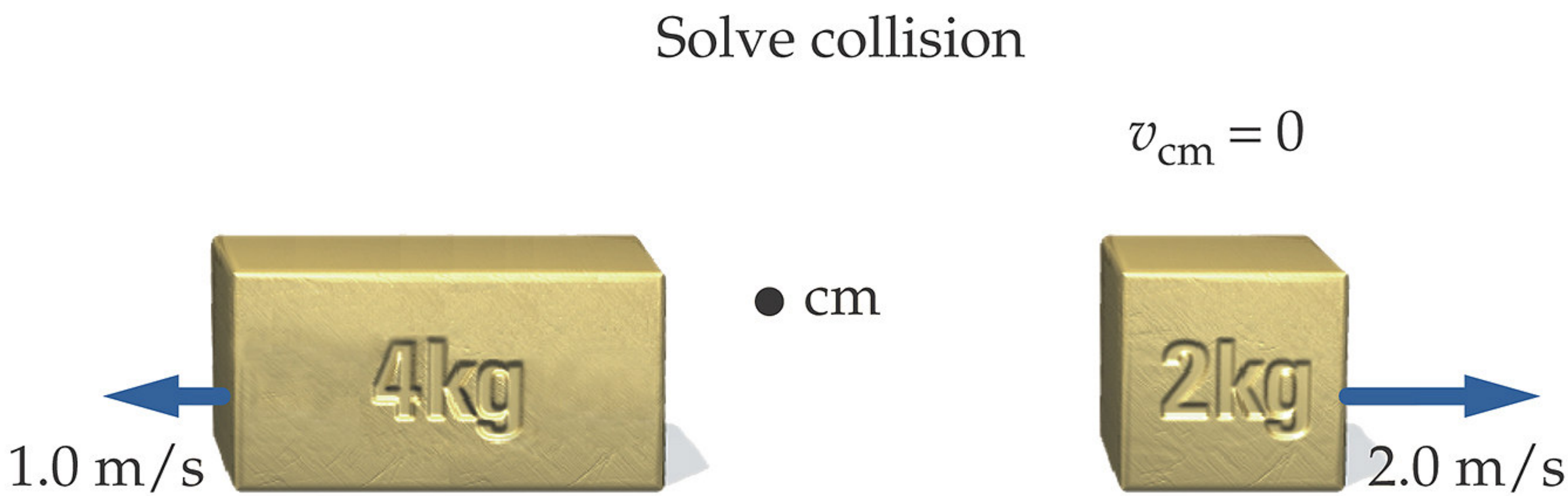
# Center of Mass Treatment



# Center of Mass Treatment

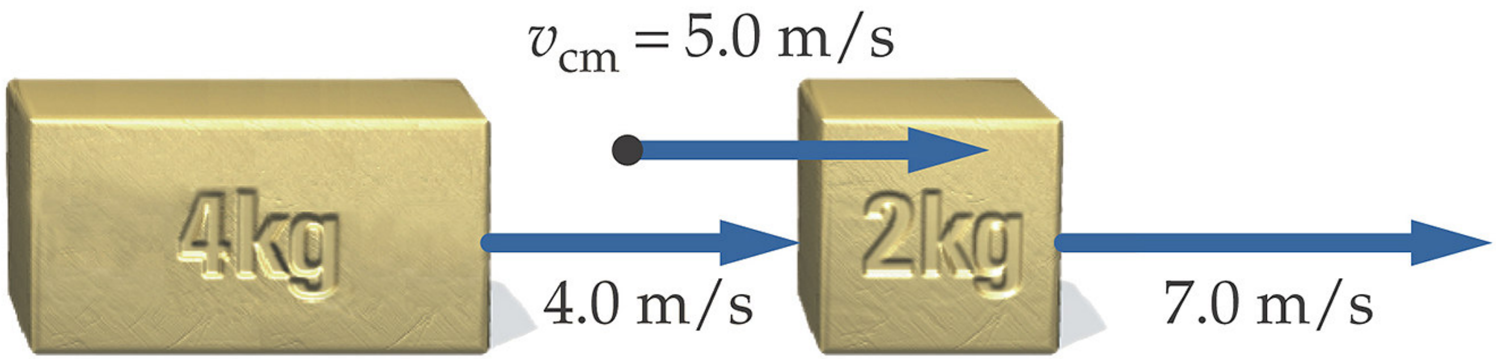


# Center of Mass Treatment

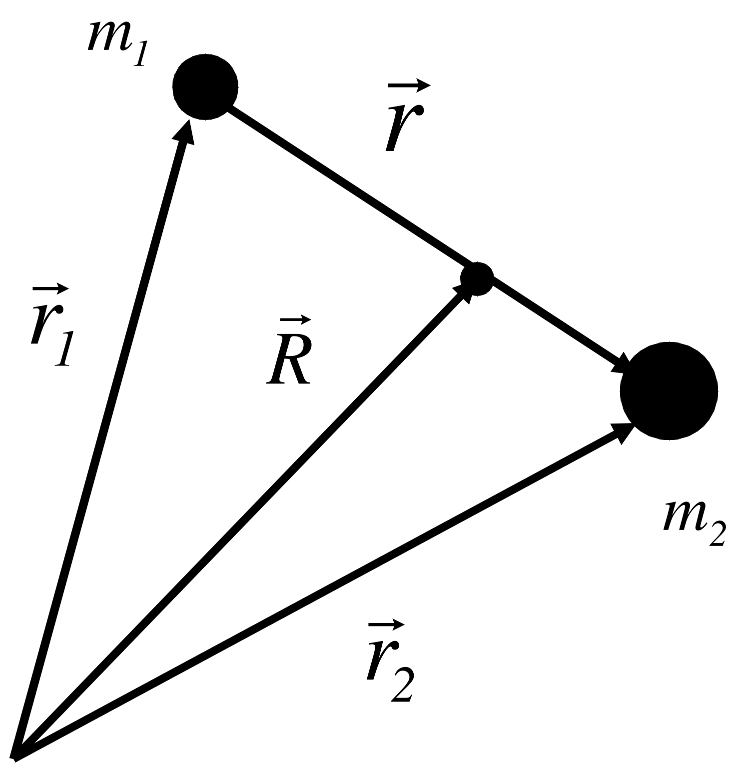


# Center of Mass Treatment

Transform back to the original frame by adding $v_{cm}$



# Center of Mass Reference System




Center of Mass System

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

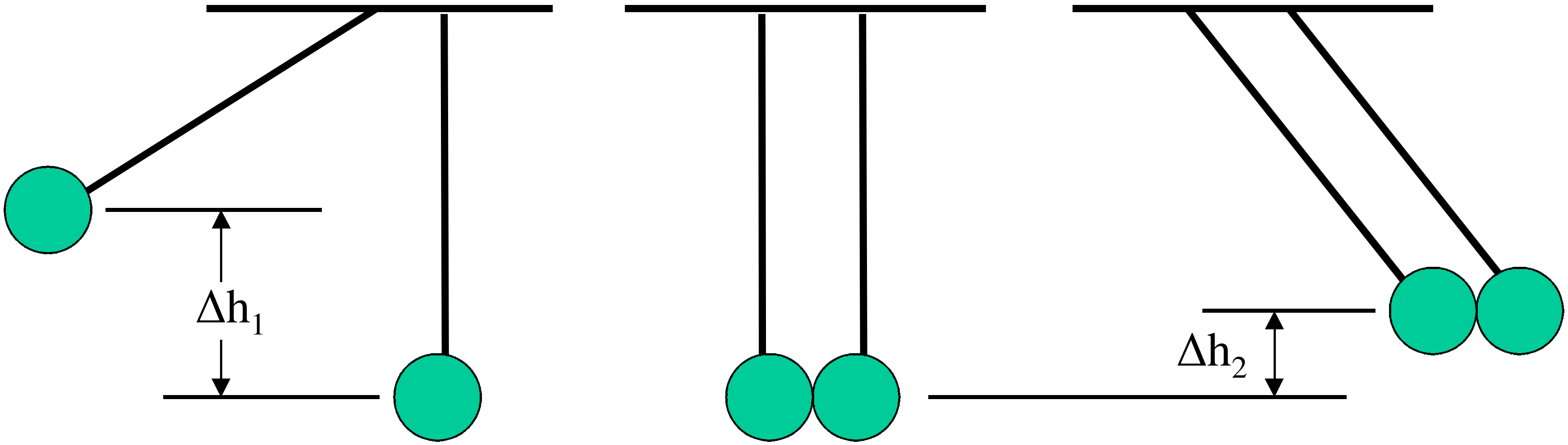
$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

Laboratory System

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r}$$

# Combining Conservation of Energy, Momentum, and KE Transfer in Equal Mass Inelastic Collisions




| Conservation of Energy | Conservation of Momentum | Conservation of Energy |
|---|---|---|
| $mg\Delta h_1 = \frac{1}{2}mv^2$ | $mv = (m+m)u$ | $\frac{1}{2}(2m)u^2 = 2mg\Delta h_2$ |
| $v = \sqrt{2g\Delta h_1}$ | $u = \frac{1}{2}v$ | $u^2 = 2g\Delta h_2$ |

$$\Delta h_2 = \frac{u^2}{2g} = \frac{v^2}{8g} = \frac{2g\Delta h_1}{8g} = \frac{\Delta h_1}{4}$$

# KE Transfer in Equal Mass Inelastic Collisions

# Conservation of Momentum

Two objects of identical mass have a collision. Initially object 1 is traveling to the right with velocity $v_1 = v_0$. Initially object 2 is at rest $v_2 = 0$.

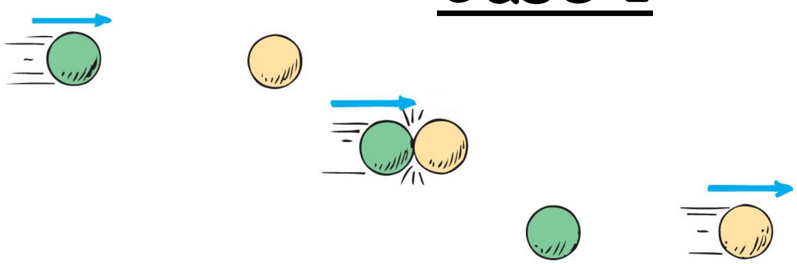
After the collision:

Case 1: $v_1' = 0$ , $v_2' = v_0$ (Elastic)

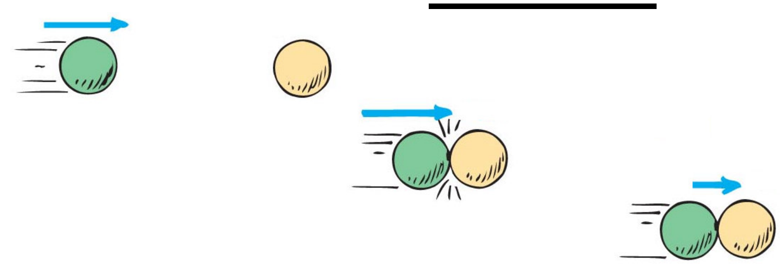
Case 2: $v_1' = ½\, v_0$ , $v_2' = ½\, v_0$ (Inelastic)

In both cases *momentum is conserved*.

Case 1

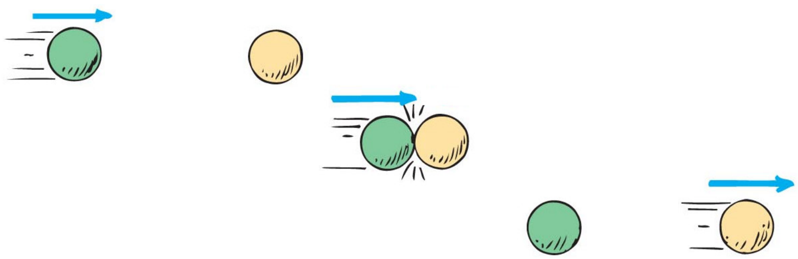


Case 2

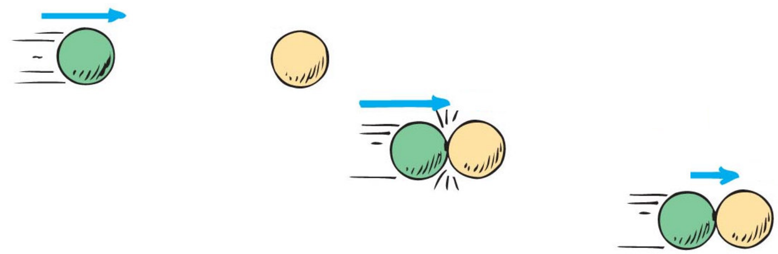


# Conservation of Kinetic Energy?

**Case 1**



**Case 2**



$KE_{After} = KE_{Before}$

(Conserved)

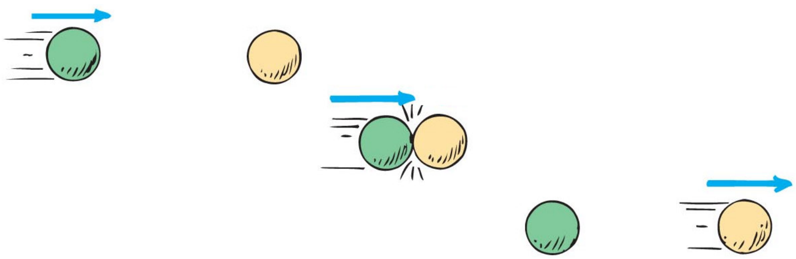
$KE_{After} = \frac{1}{2} KE_{Before}$

(Not Conserved)

$KE = \text{Kinetic Energy} = \frac{1}{2}mv_o^2$

# Where Does the KE Go?

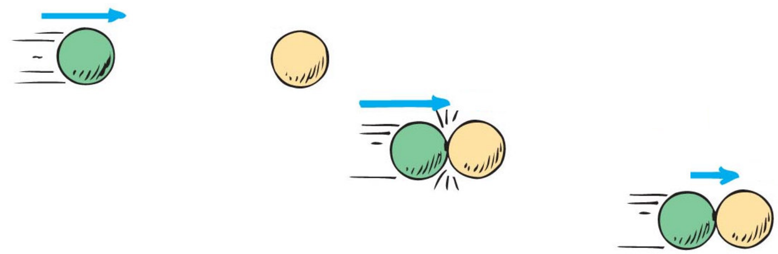
**Case 1**



$KE_{After} = KE_{Before}$

(Conserved)

**Case 2**



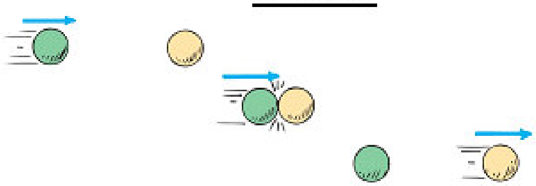
$KE_{After} = \frac{1}{2} KE_{Before}$

(Not Conserved)

In Case 2 each object shares the Total KE equally.
Therefore each object has KE = 25% of the original $KE_{Before}$

# 50% of the KE is Missing

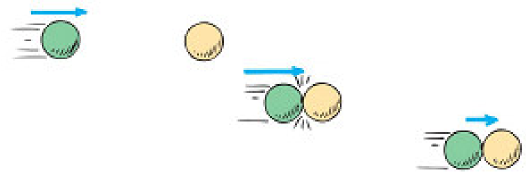
Case 1



$KE_{After} = KE_{Before}$

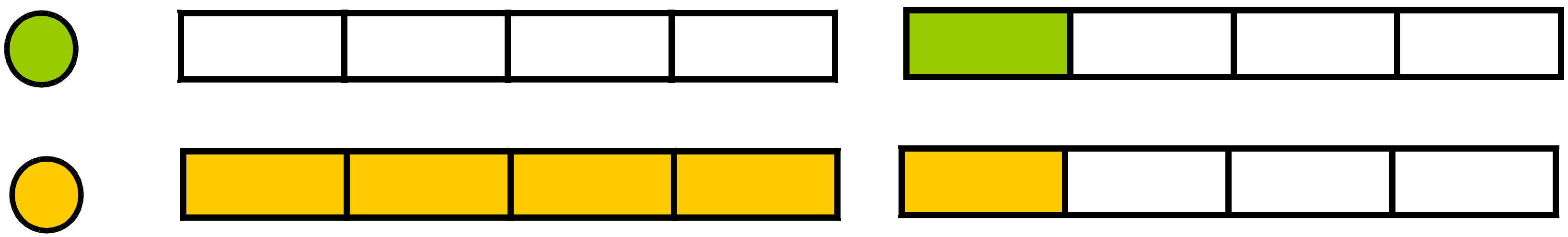
(Conserved)

Case 2



$KE_{After} = \frac{1}{2} KE_{Before}$

(Not Conserved)

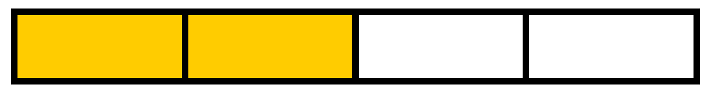


Each rectangle represents $KE = \frac{1}{4} KE_{Before}$

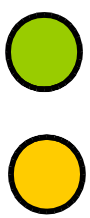
# Turn Case 1 into Case 2

Each rectangle represents KE = ¼ $KE_{Before}$

Take away 50% of KE. Now the total system KE is correct

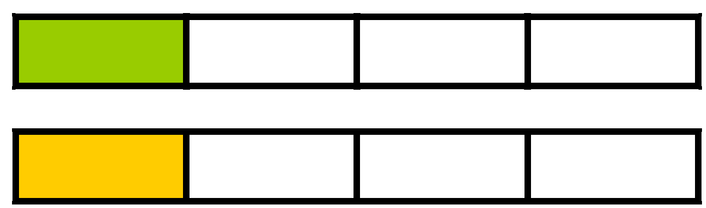


But object 2 has all the KE and object 1 has none



Use one half of the 50% taken to speed up object 1. Now it has 25% of the initial KE

Use the other half of the 50% taken to slow down object 2. Now it has only 25% of the the initial KE



Now they share the KE equally and we see where the missing 50% was spent.

# Extra Slides