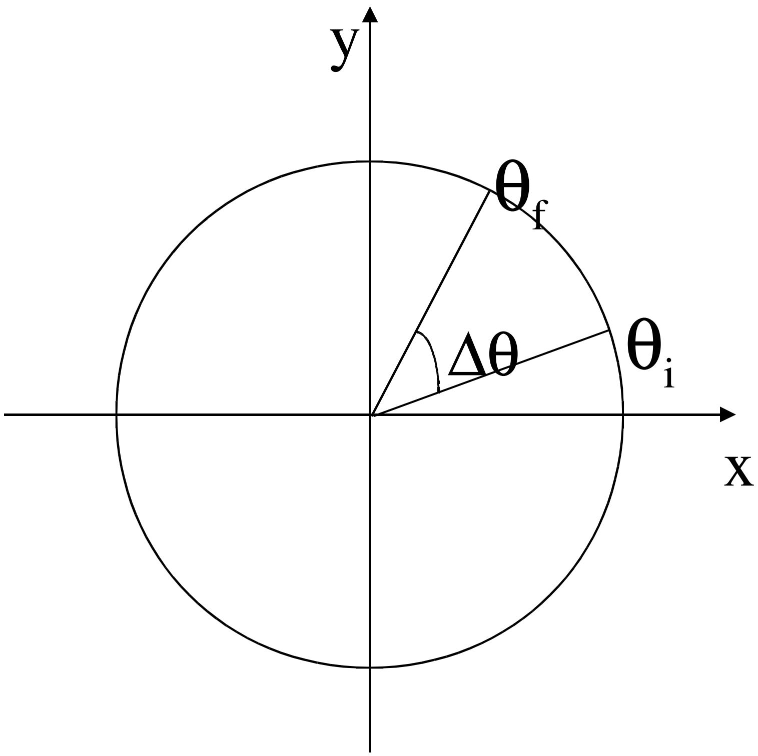
Chapter 10

Rotation

Rotation

- Rotational Kinematics: Angular velocity and Angular Acceleration
- Rotational Kinetic Energy
- Moment of Inertia
- Newton's 2nd Law for Rotation
- Applications

Angular Displacement



θ is the angular position.

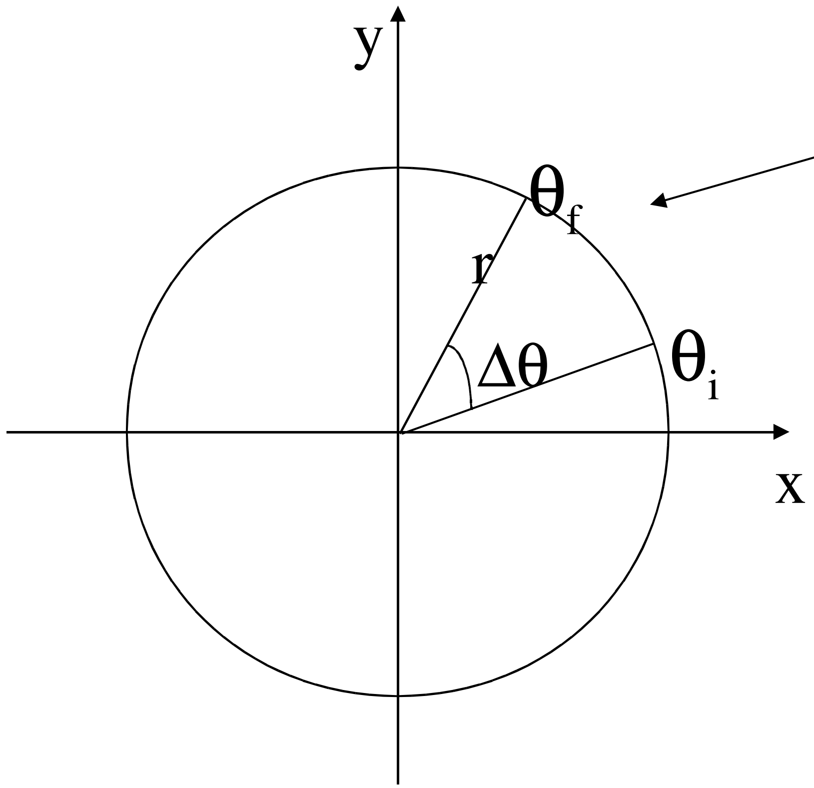
Angular displacement:

$$\Delta\theta = \theta_f - \theta_i$$

Note: angles measured CW are negative and angles measured CCW are positive. θ is measured in radians.

2π radians = 360° = 1 revolution

Arc Length



arc length = s = $r\Delta\theta$

$$\Delta\theta = \frac{s}{r}$$

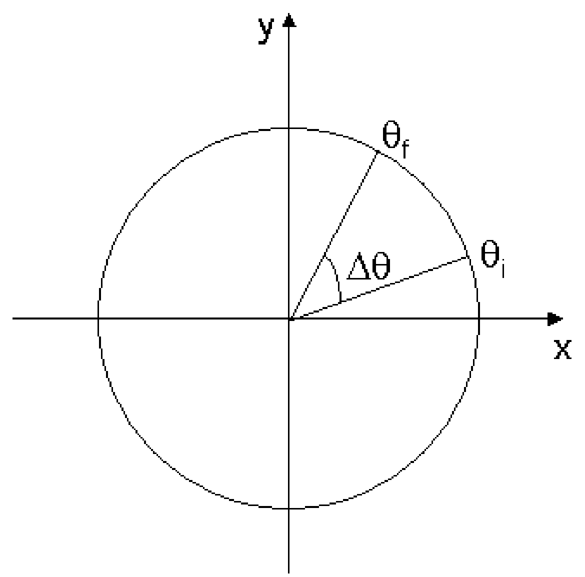
$\Delta\theta$ is a ratio of two lengths; it is a dimensionless ratio!

Angular Speed

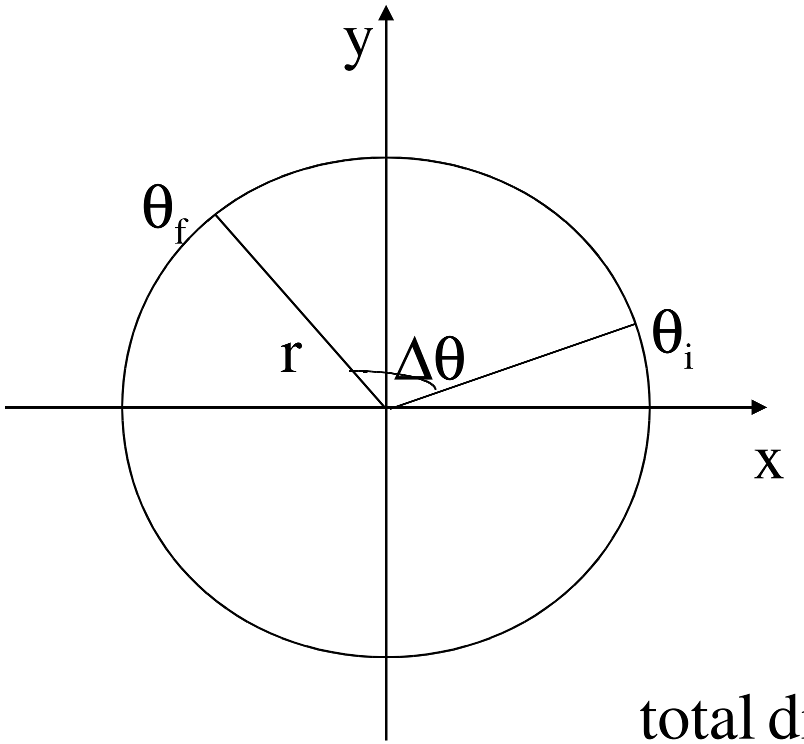
The average and instantaneous angular velocities are:

$$\omega_{\text{av}} = \frac{\Delta\theta}{\Delta t} \quad \text{and} \quad \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

ω is measured in rads/sec.



Angular Speed



An object moves along a circular path of radius r; what is its average speed?

$$v_{av} = \frac{\text{total distance}}{\text{total time}} = \frac{r\Delta\theta}{\Delta t} = r\left(\frac{\Delta\theta}{\Delta t}\right) = r\omega_{av}$$

Also, $v = r\omega$ (instantaneous values).

Period and Frequency

The time it takes to go one time around a closed path is called the **period** (T).

$$v_{\text{av}} = \frac{\text{total distance}}{\text{total time}} = \frac{2\pi r}{T}$$

Comparing to v = rω: $\omega = \frac{2\pi}{T} = 2\pi f$

f is called the **frequency**, the number of revolutions (or cycles) per second.

Comparison of Kinematic Equations

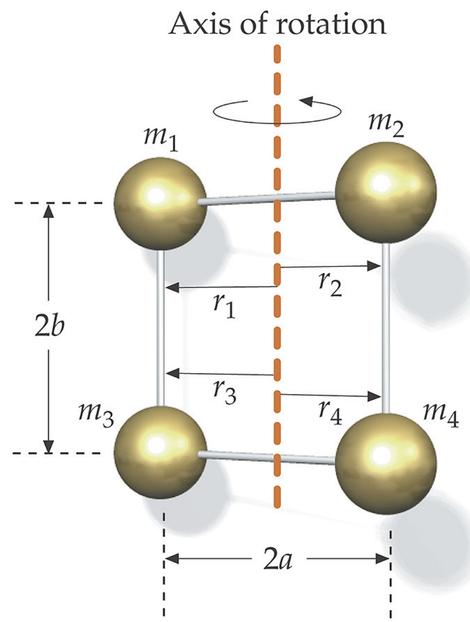
Angular		Linear
$\Delta\theta$	Displacement	Δx
$\omega = \dfrac{d\theta}{dt}$	Velocity	$v_x = \dfrac{dx}{dt}$
$\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$	Acceleration	$a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}$
$\omega = \omega_0 + \alpha t$	Velocity	$v_x = v_{0x} + a_x t$
$\Delta\theta = \omega_{\text{av}}\,\Delta \text{t}$		$\Delta x = v_{\text{av}\,x}\,\Delta t$
$\omega_{\text{av}} = \frac{1}{2}(\omega_0 + \omega)$		$v_{\text{av}\,x} = \frac{1}{2}(v_{0x} + v_x)$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$		$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$
$\omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta$		$v_x^2 = v_{0x}^2 + 2a_x\,\Delta x$

Rotational Kinetic Energy

$$KE_R = \tfrac{1}{2} I\omega^2 = \sum_i \tfrac{1}{2} m_i v_i^2$$

$$\sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2}\ m_i \left(r_i\ \omega\right)^2 = \sum_i \tfrac{1}{2} m_i r_i^2 \omega^2$$

$$= \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$

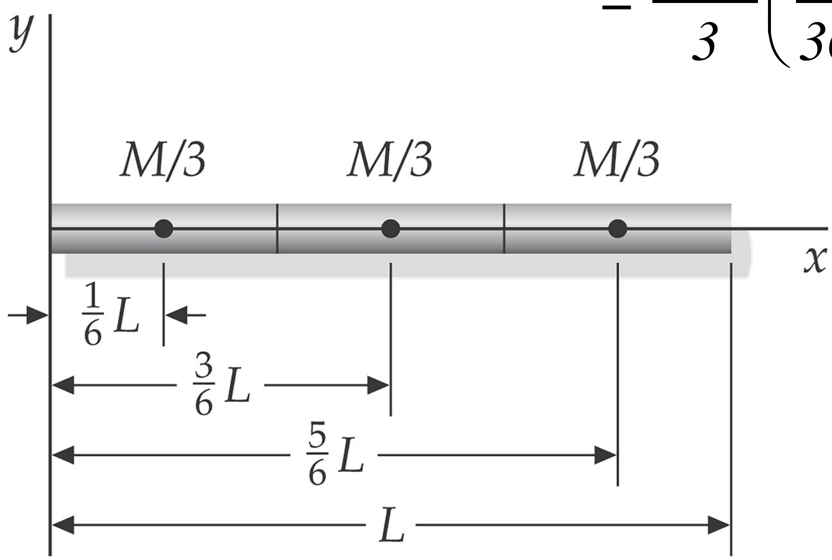


Estimating the Moment of Inertia

Point particle assumption. The particle is at the center of mass of each rod segment.

$$I = \sum ML^2 = \frac{ML^2}{3}\left(\left(\frac{1}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{5}{6}\right)^2\right)$$

$$= \frac{ML^2}{3}\left(\frac{35}{36}\right) = \frac{35ML^2}{108} = 0.324ML^2$$



The actual value is

I = (1/3) ML^2

Moment of Inertia: Thin Uniform Rod

A favorite application from calculus class.

$$I = \int x^2 dm = \int x^2 \lambda dx = \int \; x^2 \frac{M}{L} \; dx = \frac{M}{L} \int_0^L x^2 \; dx$$

$$I = \frac{1}{3} \frac{M}{L} L^3 = \frac{1}{3} ML^2$$

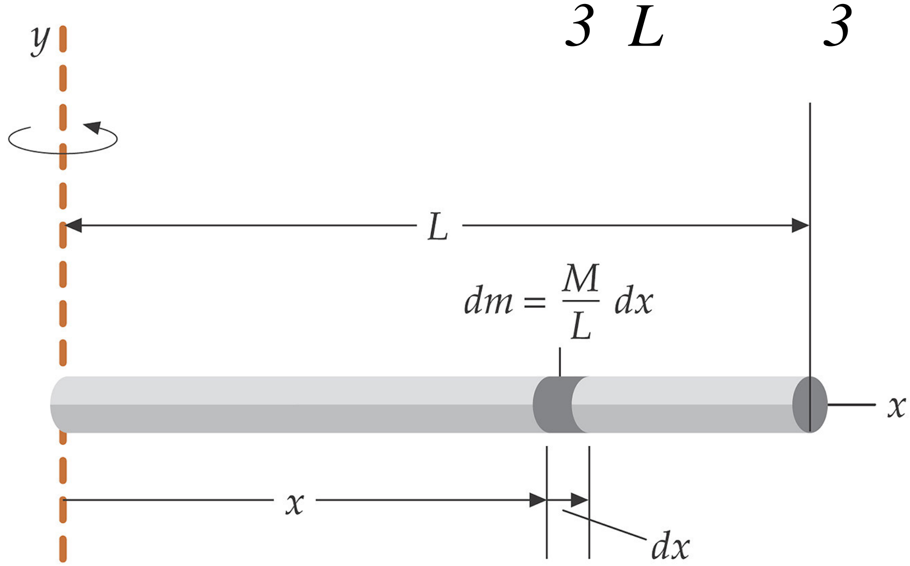
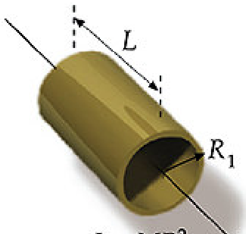


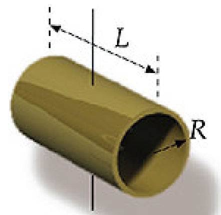
Table 9-1 Moments of Inertia of Uniform Bodies of Various Shapes

Thin cylindrical shell about axis



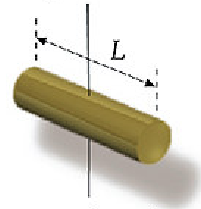
$I = MR^2$

Thin cylindrical shell about diameter through center



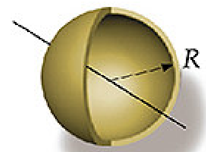
$I = \frac{1}{2}MR^2 + \frac{1}{12}ML^2$

Thin rod about perpendicular line through center



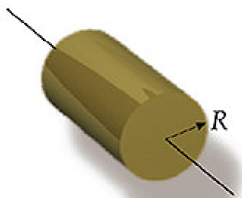
$I = \frac{1}{12}ML^2$

Thin spherical shell about diameter



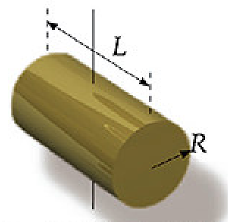
$I = \frac{2}{3}MR^2$

Solid cylinder about axis



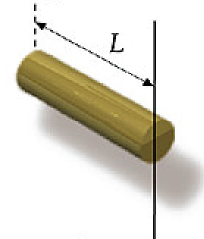
$I = \frac{1}{2}MR^2$

Solid cylinder about diameter through center



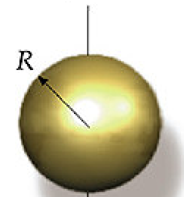
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$

Thin rod about perpendicular line through one end



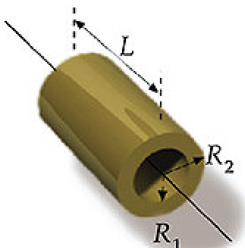
$I = \frac{1}{3}ML^2$

Solid sphere about diameter



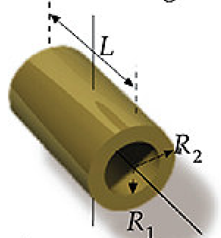
$I = \frac{2}{5}MR^2$

Hollow cylinder about axis



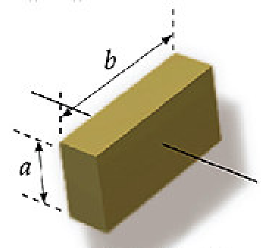
$I = \frac{1}{2}M(R_1^2 + R_2^2)$

Hollow cylinder about diameter through center



$I = \frac{1}{4}M(R_1^2 + R_2^2) + \frac{1}{12}ML^2$

Solid rectangular parallelepiped about axis through center perpendicular to face

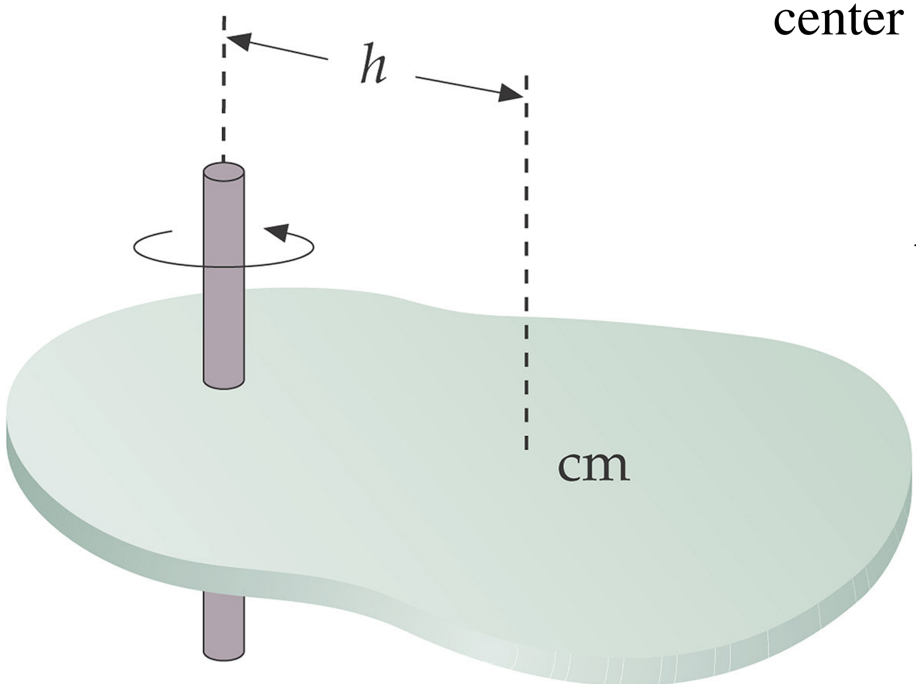


$I = \frac{1}{12}M(a^2 + b^2)$

*A disk is a cylinder whose length L is negligible. By setting $L = 0$, the above formulas for cylinders hold for disks.

Parallel Axis Theorem

The Parallel Axis theorem is used with the center of mass moments of inertia found in the table to extend those formulas to non-center of mass applications



$$I = I_{cm} + Mh^2$$

Application of the Parallel Axis Theorem

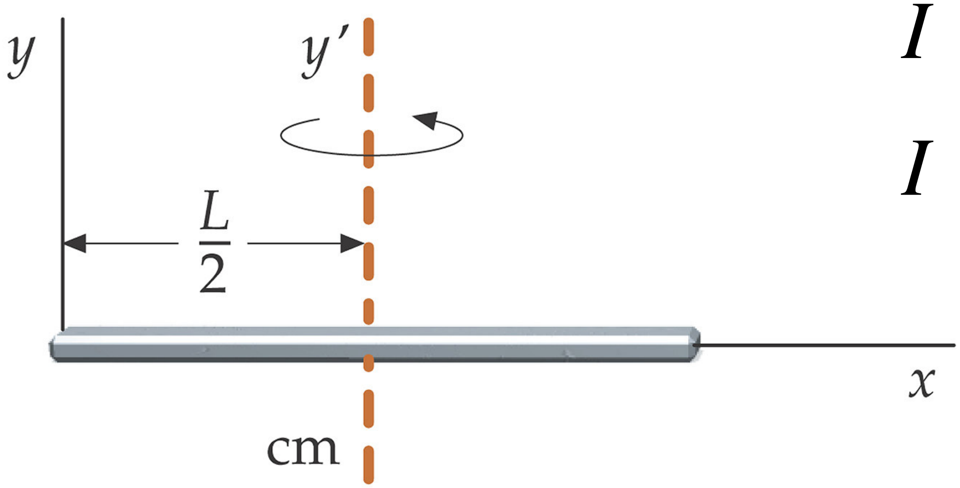
$$I_{cm} = \tfrac{1}{12} ML^2$$

$$I = I_{cm} + Mh^2$$

$$I = I_{cm} + M\left(\frac{L}{2}\right)^2$$

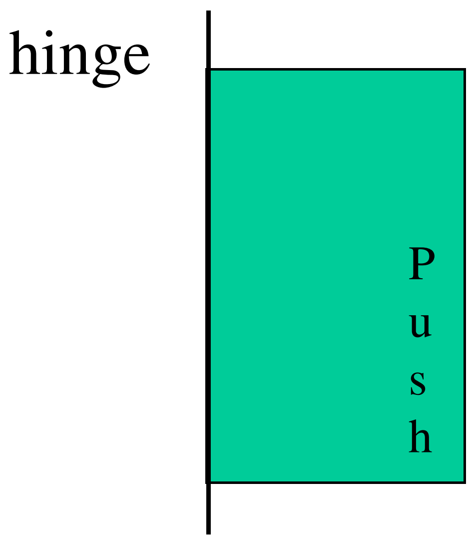
$$I = \tfrac{1}{12} ML^2 + \tfrac{1}{4} ML^2$$

$$I = \tfrac{1}{3} ML^2$$



Torque

A torque is caused by the application of a force, on an object, at a point other than its center of mass or its pivot point.



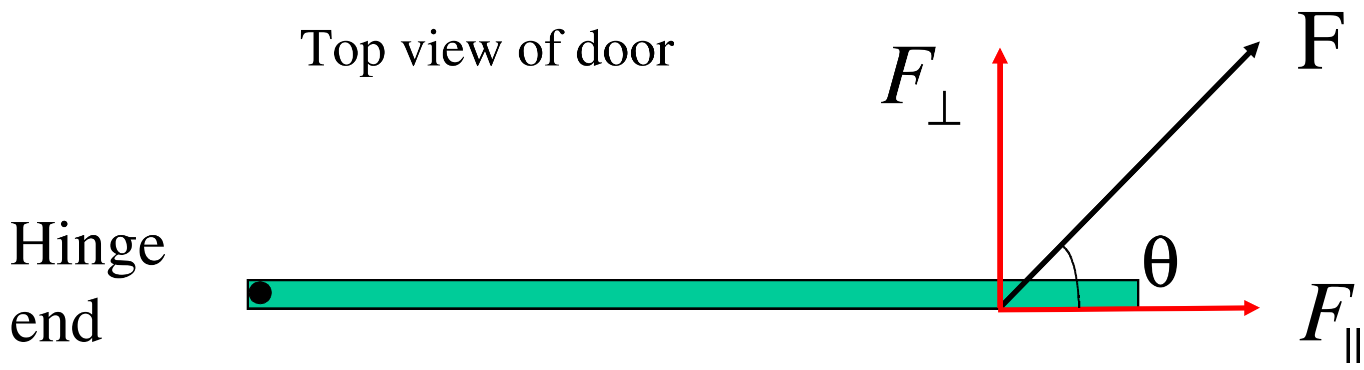
Q: Where on a door do you normally push to open it?

A: Away from the hinge.

A rotating (spinning) body will continue to rotate unless it is acted upon by a torque.

Torque

Torque method 1:



$$\tau = rF_{\perp}$$

r = the distance from the rotation axis (hinge) to the point where the force F is applied.

$F_{\perp}$ is the component of the force F that is perpendicular to the door (here it is Fsinθ).

Torque

The units of torque are Newton-meters (Nm)
(not joules!).

By convention:

- When the applied force causes the object to rotate counterclockwise (CCW) then τ is positive.
- When the applied force causes the object to rotate clockwise (CW) then τ is negative.

Be careful with placement of force vector components.

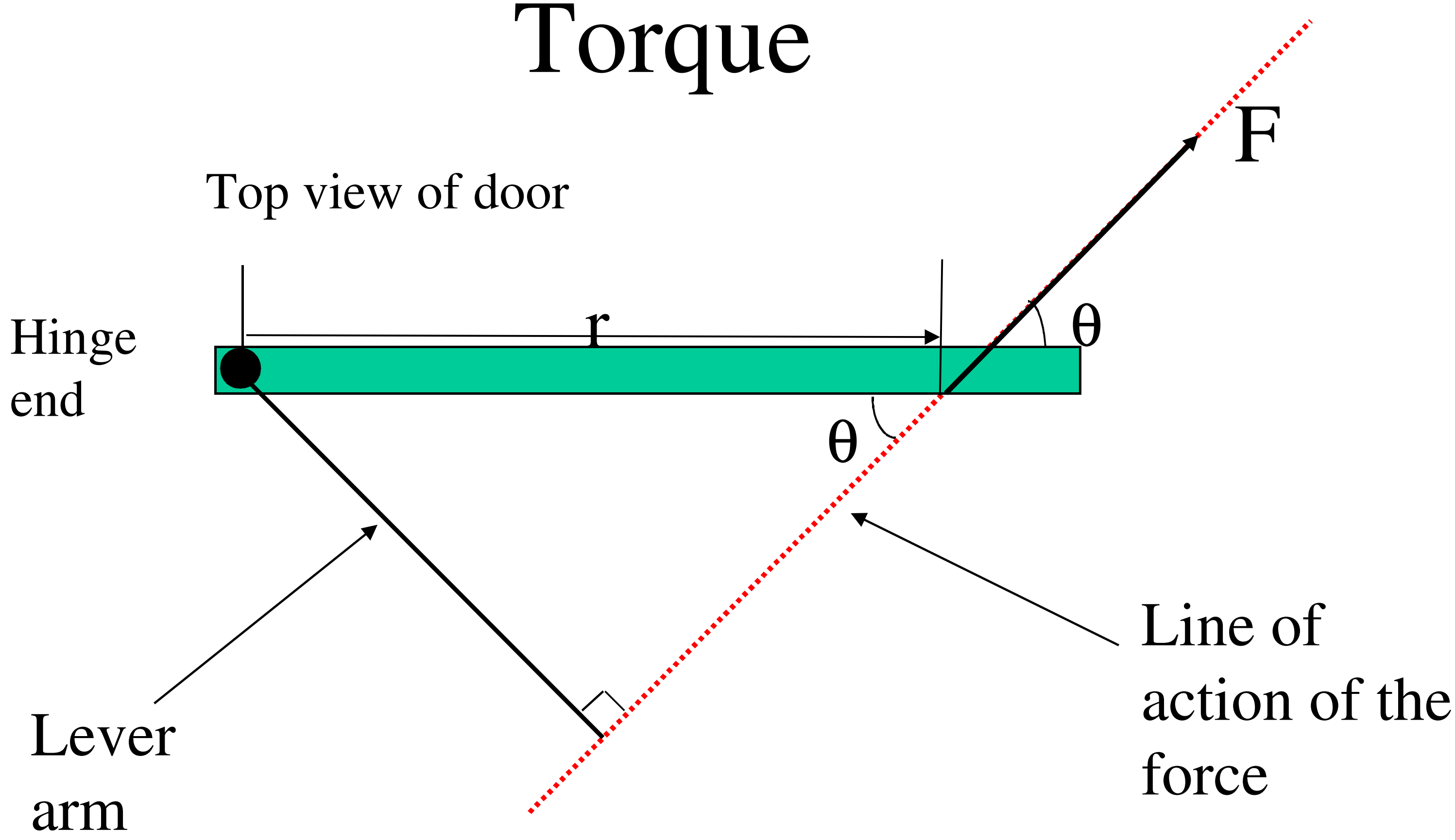
Torque

Torque method 2: $$\tau = r_{\perp} F$$

$r_{\perp}$ is called the lever arm and F is the magnitude of the applied force.

Lever arm is the perpendicular distance to the line of action of the force from the pivot point or the axis of rotation.

Torque



$$\sin\theta = \frac{r_\perp}{r}$$

$$r_\perp = r\sin\theta$$

The torque is: $\tau = r_\perp F$

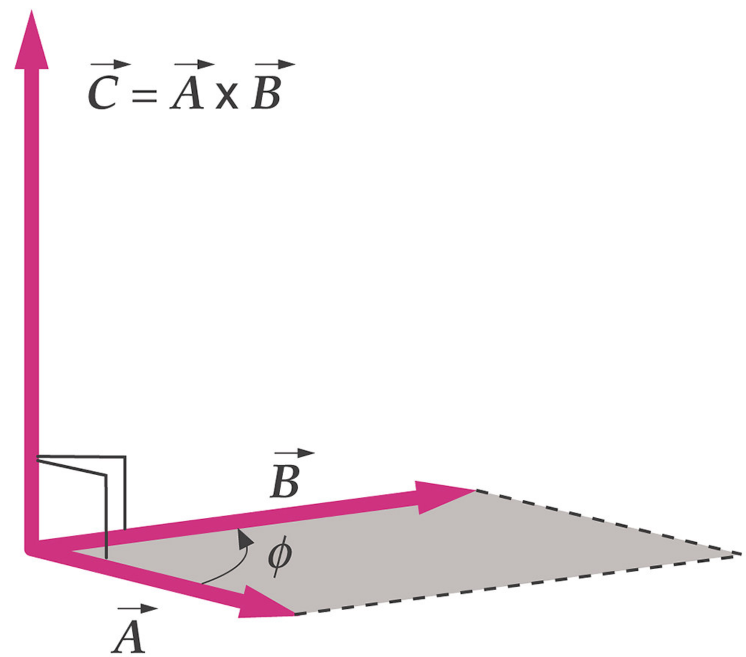
$= rF\sin\theta$

Same as before

The Vector Cross Product

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}|\left|\vec{F}\right| sin\theta = rFsin\theta$$



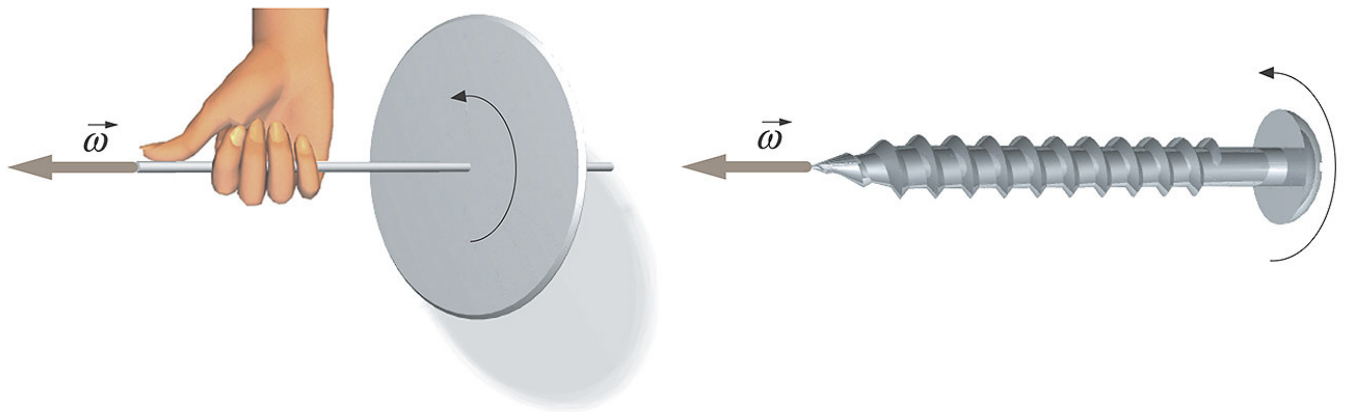
The magnitude of C

C = ABsin(Φ)

The direction of C is perpendicular to the plane of A and B.

Physically it means the product of A and the portion of B that is perpendicular to A.

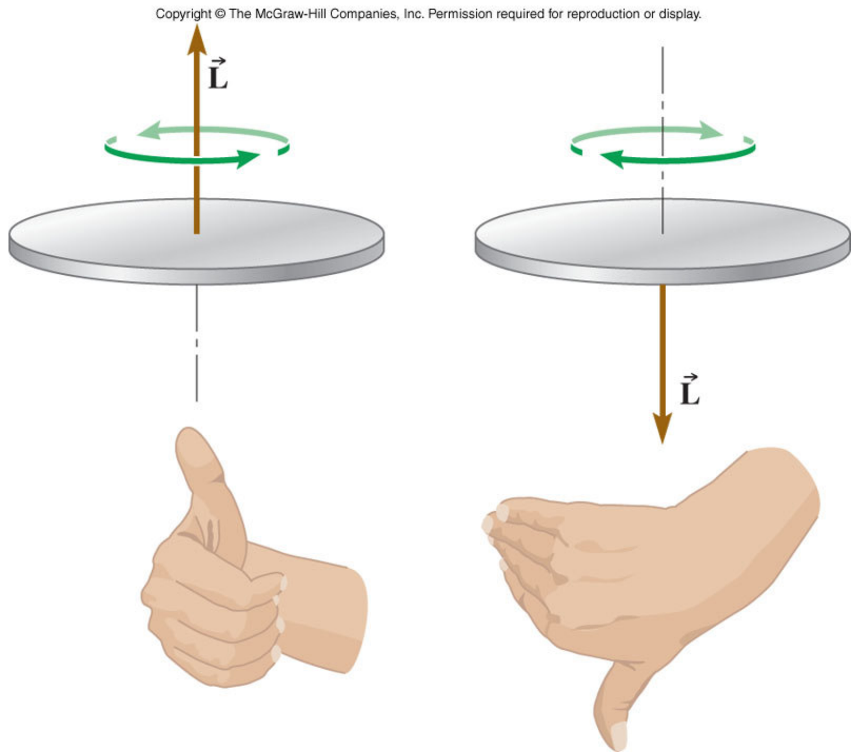
Vector Nature of the Cross Product



In rotation the direction of the vectors can be determined by using the right-hand rule

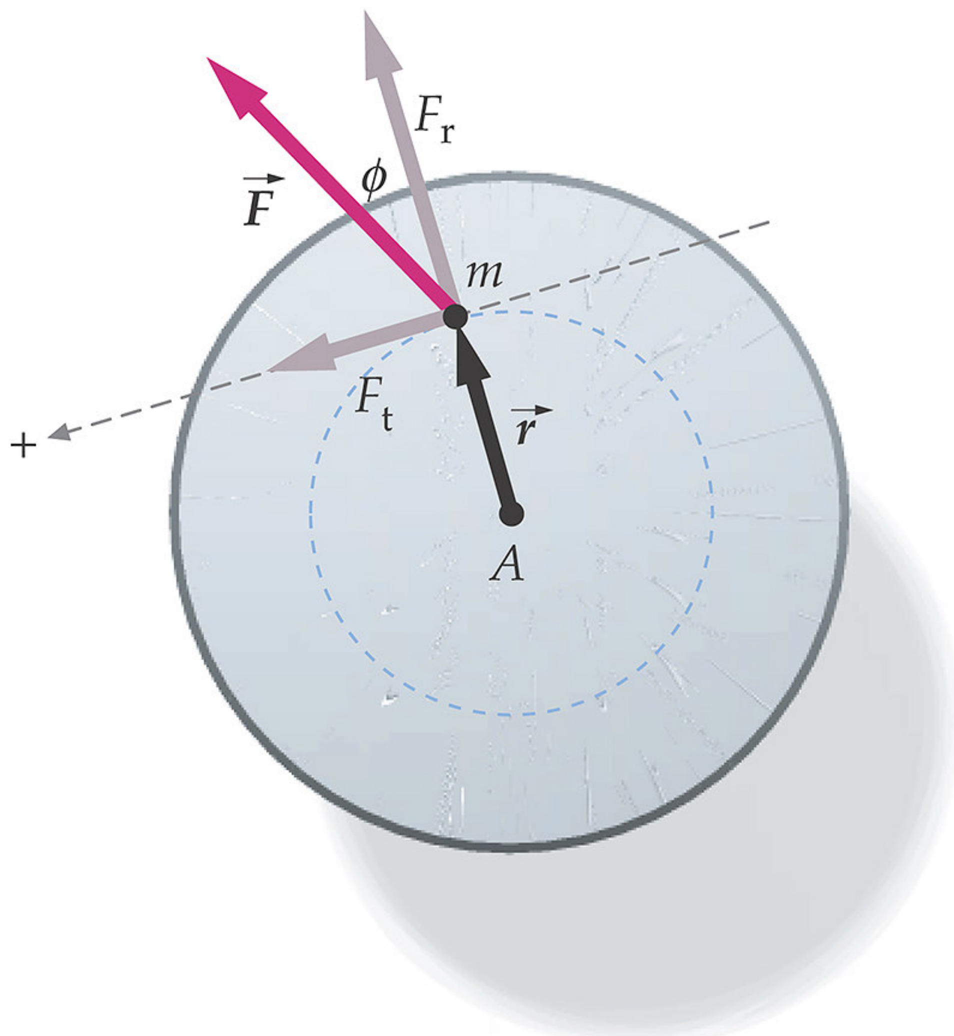
The Right-Hand Rule

Curl the fingers of your right hand so that they curl in the direction a point on the object moves, and your thumb will point in the direction of the angular momentum.



Torque is an example of a vector cross product

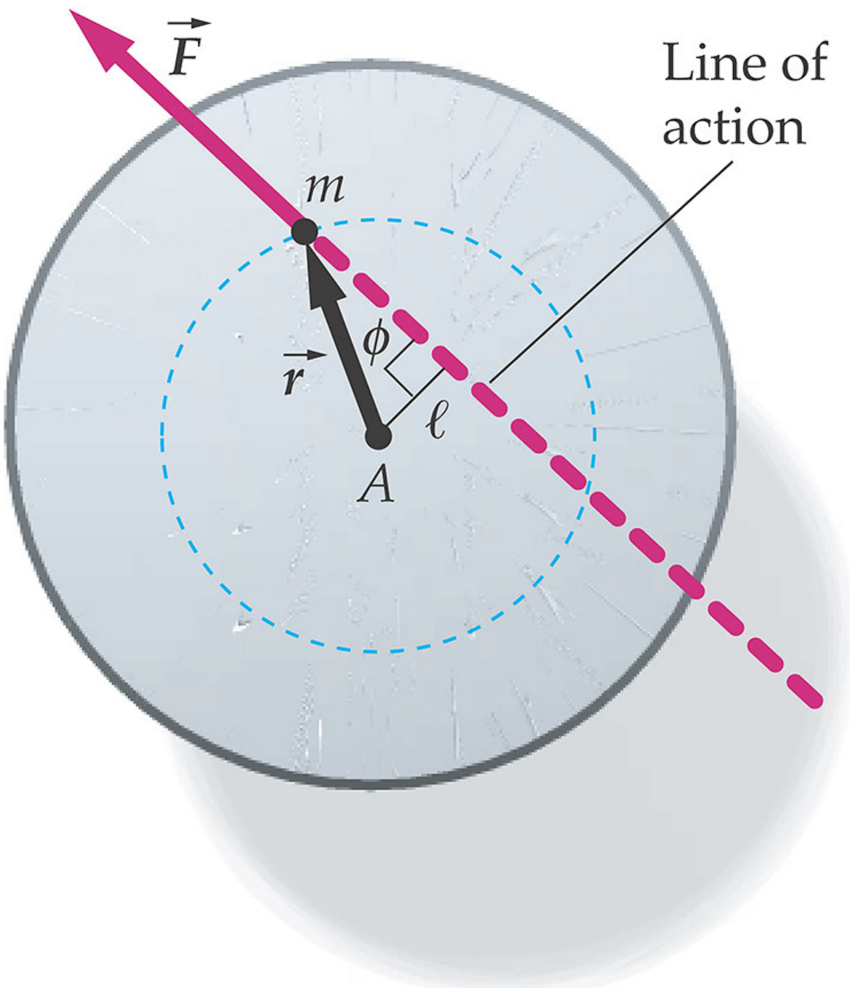
Torque (Disk) - F_t - Component



The radial component F_r <u>cannot</u> cause a rotation.

Only the tangential component, F_t can cause a rotation.

Torque (Disk) - Lever Arm Method



Sometimes the Lever Arm method is easier to implement, especially if there are several force vectors involved in the problem.

Torque Examples

Hanging Sign

Meter stick balanced at 25cm.

A Winch and A Bucket

Tension in String - Massive Pulley

Atwood with Massive Pulley

Two Blocks and a Pulley

Bowling Ball

Loop, Disk, Sphere and Block

Equilibrium

The conditions for equilibrium are:

$$\sum \vec{\mathbf{F}} = 0$$

Linear motion

$$\sum \boldsymbol{\tau} = 0$$

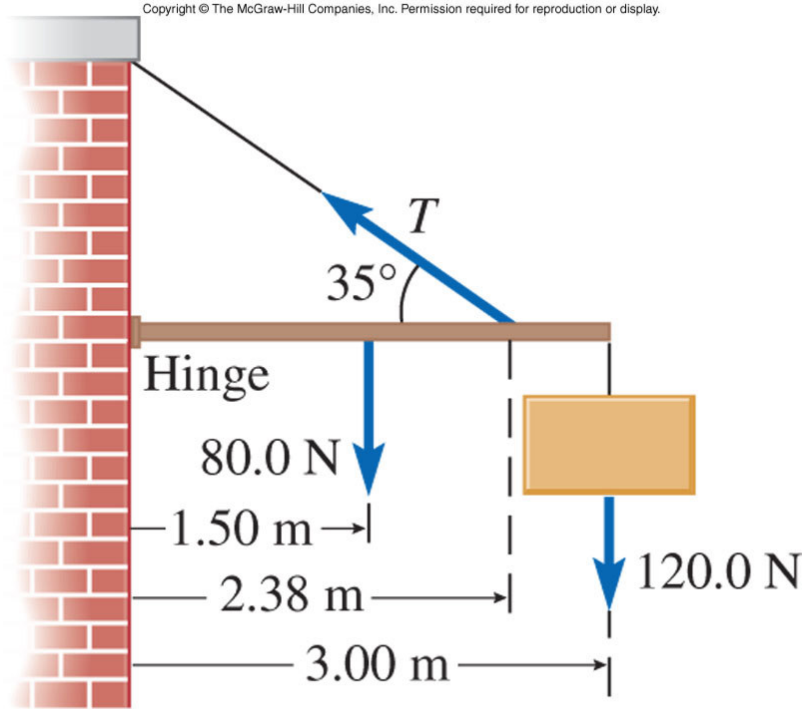
Rotational motion

For motion in a plane we now have three equations to satisfy.

$$\sum F_x = \sum F_y = 0; \quad \sum \tau_z = 0$$

Using Torque

A sign is supported by a uniform horizontal boom of length 3.00 m and weight 80.0 N. A cable, inclined at a 35° angle with the boom, is attached at a distance of 2.38 m from the hinge at the wall. The weight of the sign is 120.0 N.

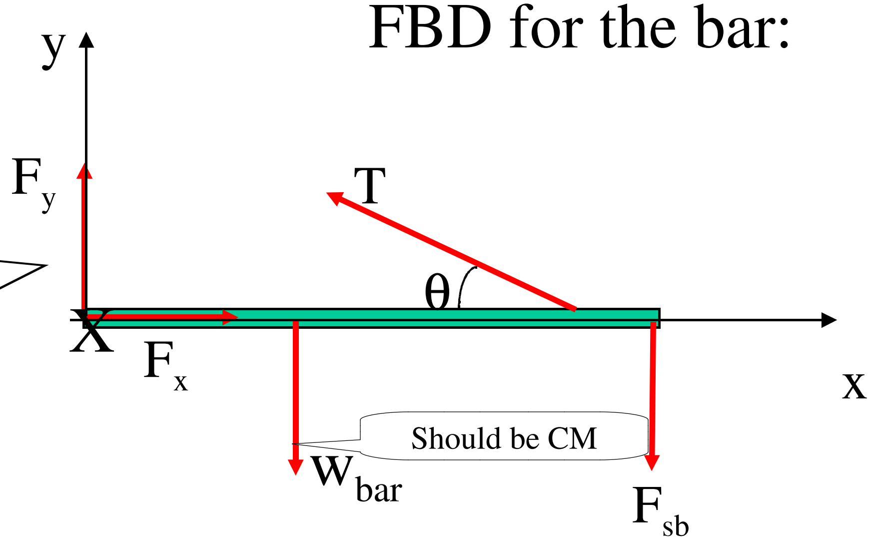


What is the tension in the cable and what are the horizontal and vertical forces exerted on the boom by the hinge?

Using Torque

This is important!
You need two components for F, not just the expected perpendicular normal force.

FBD for the bar:



Apply the conditions for equilibrium to the bar:

$$(1)\ \sum F_x = F_x - T\cos\theta = 0$$

$$(2)\ \sum F_y = F_y - w_{\text{bar}} - F_{sb} + T\sin\theta = 0$$

$$(3)\ \sum \tau = -w_{\text{bar}}\left(\frac{L}{2}\right) - F_{sb}(L) + (T\sin\theta)x = 0$$

Using Torque

Equation (3) can be solved for T:

$$T = \frac{w_{\text{bar}}\left(\frac{L}{2}\right) + F_{sb}(L)}{x\sin\theta}$$
$$= 352\text{ N}$$

Equation (1) can be solved for F_x:

$$F_x = T\cos\theta = 288\text{ N}$$

Equation (2) can be solved for F_y:

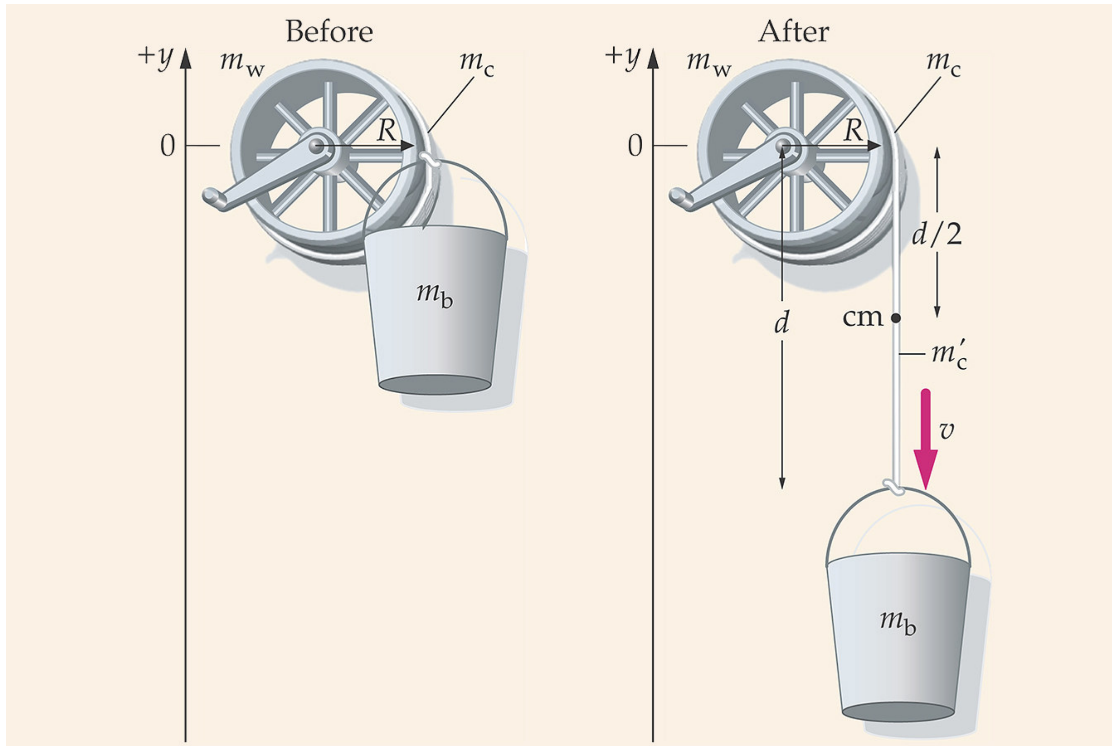
$$F_y = w_{\text{bar}} + F_{sb} - T\sin\theta$$
$$= -2.00\text{ N}$$

A Winch and a Bucket

Finally a massive pulley and massive string problem.

The pulley has its mass concentrated in the rim. Consider it to be a loop for moment of inertia considerations.

This example in the book takes a couple of short cuts that are not obvious.

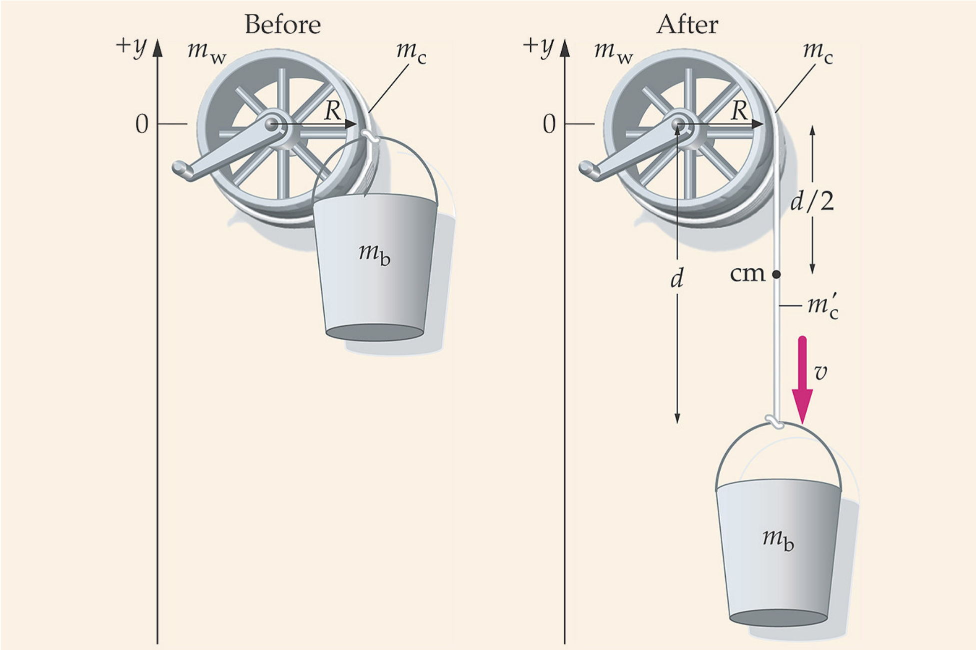


A Winch and a Bucket

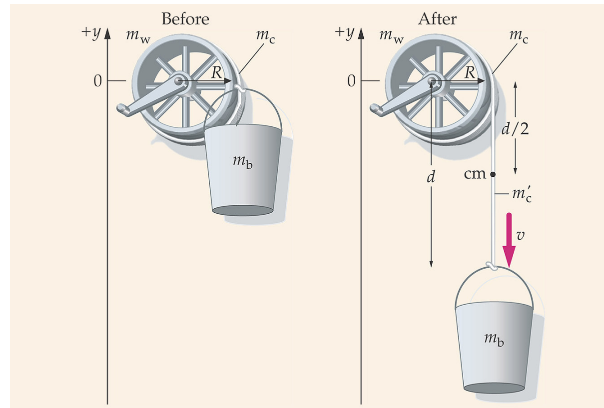
The origin of the y-axis is placed at the level of the axis of the winch.

The initial level of the handle of the bucket is also placed at the origin.

This simplifies distance measurements for figuring the gravitational potential energy.



A Winch and a Bucket



Tricks -

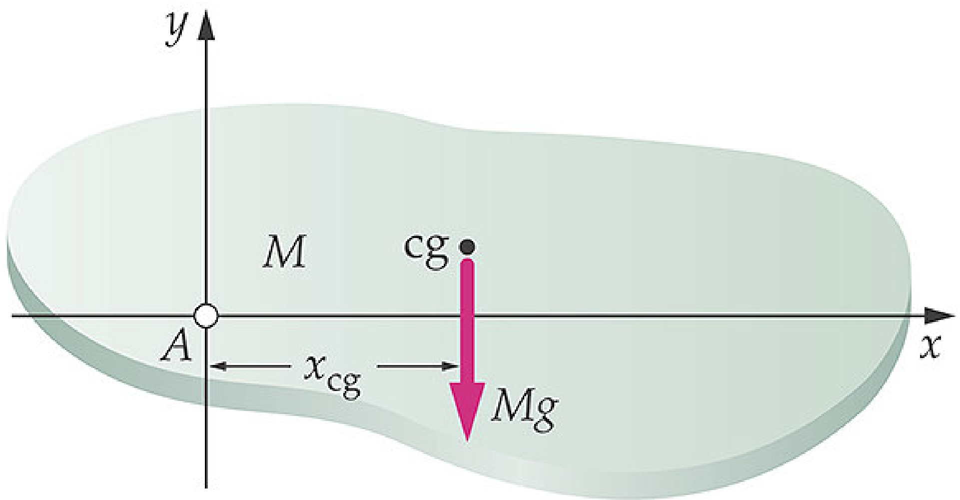
For a hoop or any other object similar in construction all the mass is concentrated along the outside edge. For these objects the rotational KE and the linear KE are identical in form.

Therefore the rotational KE of the winch and the cable on the winch are represented by their linear forms and no details are given.

$$KE_R = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2 = \tfrac{1}{2} mv^2 = KE$$

The cable on the winch and the portion falling are combined together in one KE term without a word of explanation.

Torque Due to Gravity



This is a thin 2 dimensional object in the x-y plane.

Assume that gravity acts at the center of mass.

This can be used to determine the CM of a thin irregularly shaped object

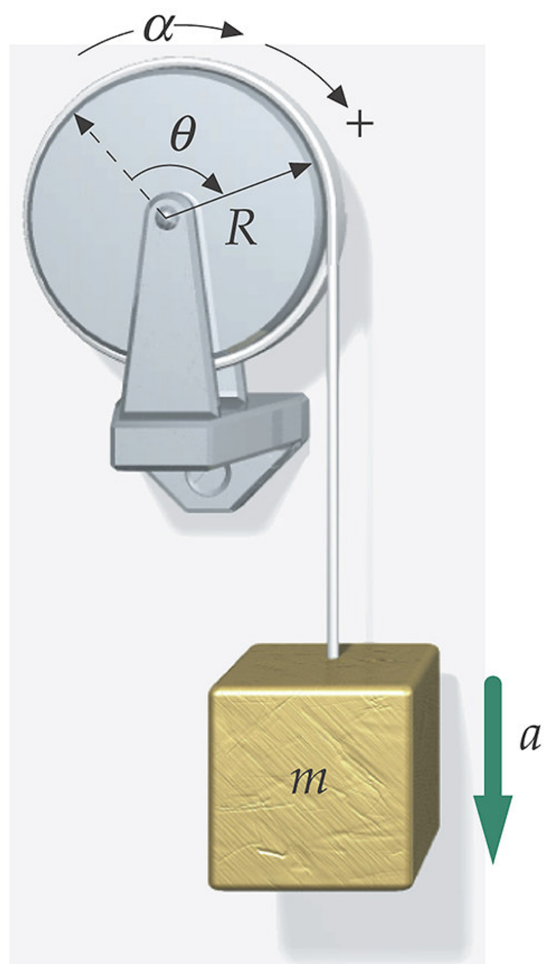
Tension in a String - Massive Pulley

Massive pulley- moment of inertia needed.

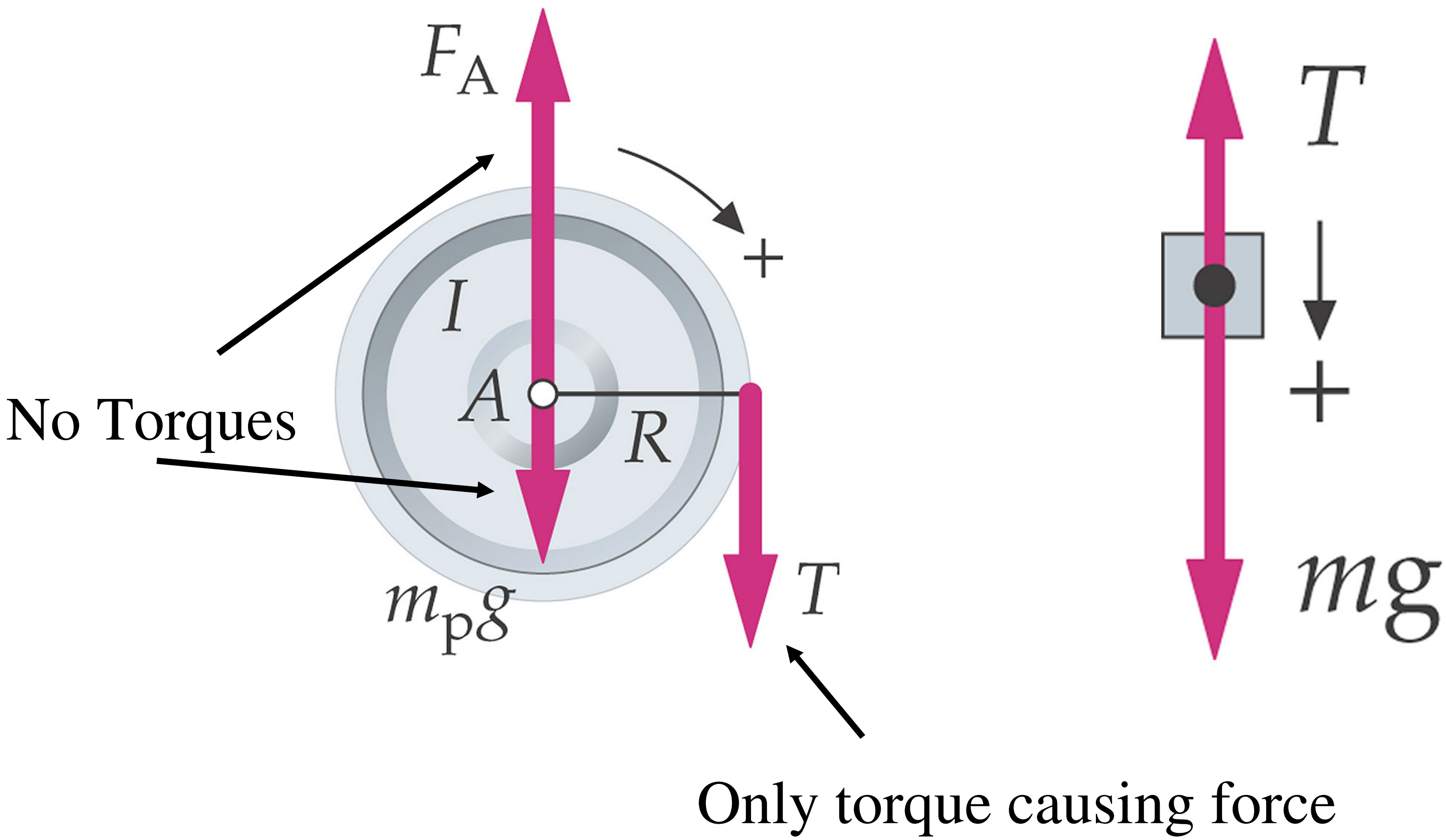
Wheel bearing frictionless

Non-slip condition

Massless rope



Tension in a String - Massive Pulley



Tension in a String - Massive Pulley

Solve by applying Newton's Laws

$$\tau = TR = I\alpha \qquad (1)$$

$$\sum F_y = -T + mg = ma \qquad (2)$$

$$a = R\alpha \qquad (3)$$

$$a = \frac{mg - T}{m}$$

$$a = \frac{g}{1 + \frac{I}{mR^2}}$$

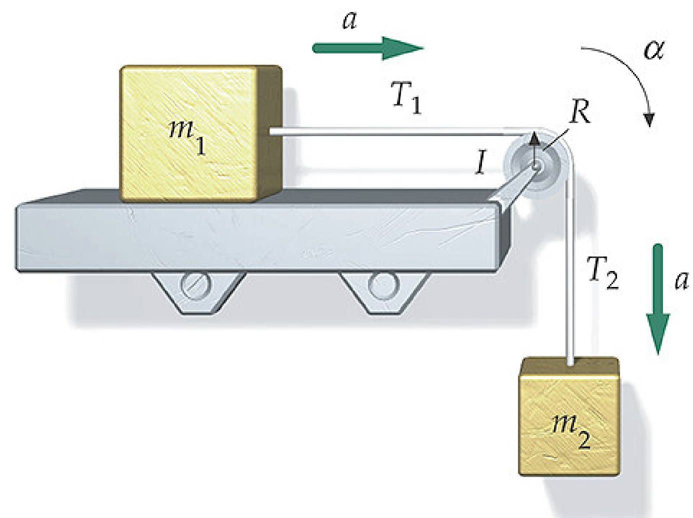
$$\frac{mg - T}{m} = R\left(\frac{TR}{I}\right) = \frac{R^2}{I}T$$

$$T = \frac{mg}{1 + \frac{mR^2}{I}}$$

Solve (2) for **a** and solve (1) for **α** and then use (3)

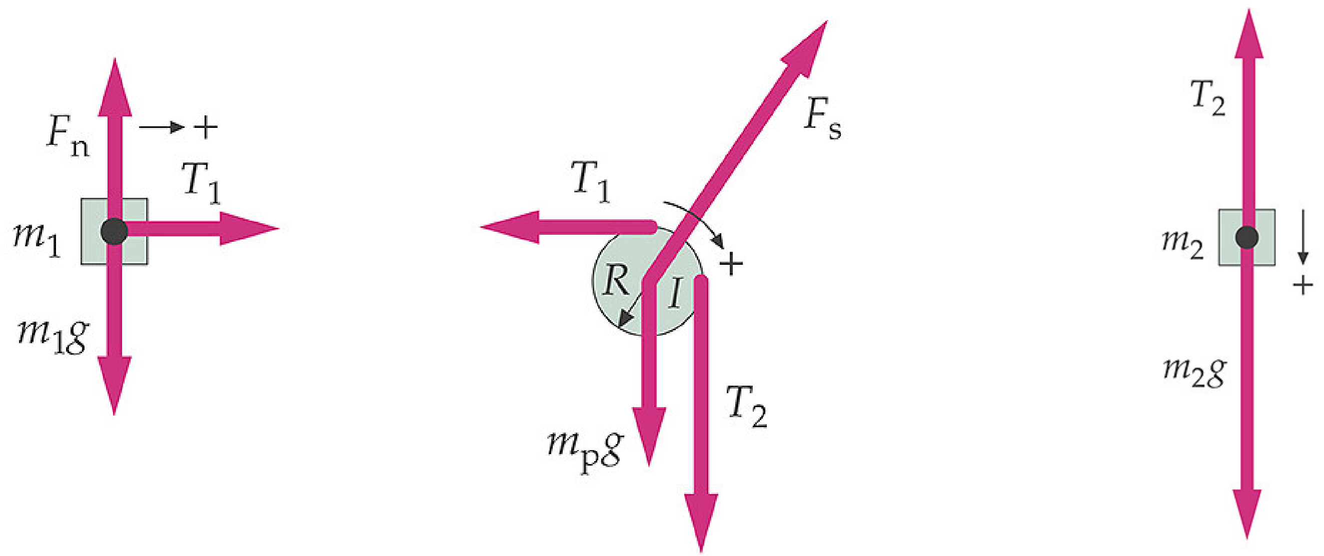
Two Blocks and a Pulley - II

- Frictionless surface
- Massive pulley - Need the moment of inertia
- Tension not continuous across pulley - because pulley has mass.
- Friction on pulley - string does not slip. - Non-slip condition $a_t = R\alpha$
- Massless string



Two Blocks and a Pulley - II

F_s is the pulley reaction to forces acting on it for the purpose of maintaining the equilibrium of the pulley center of mass.



We include F_s and m_pg in our initial discussions of this system. In subsequent problem solving there is no need to include them.

Two Blocks and a Pulley - II

$$\left.\begin{aligned} &T_1 = m_1 a;\; m_2 g - T_2 = m_2 a \\ &T_2 R - T_1 R = I\alpha \\ &a = R\alpha \end{aligned}\right\}$$

$$a = \frac{m_2 g}{m_1 + m_2 + \left(\frac{I}{R}\right)^2}$$

$$T_1 = \frac{m_2}{m_1 + m_2 + \left(\frac{I}{R}\right)^2} m_1 g$$

$$T_2 = \frac{m_1 + \left(\frac{I}{R}\right)^2}{m_1 + m_2 + \left(\frac{I}{R}\right)^2} m_2 g$$

If I = 0, the case of the massless pulley

$$a = \frac{m_2 g}{m_1 + m_2}$$

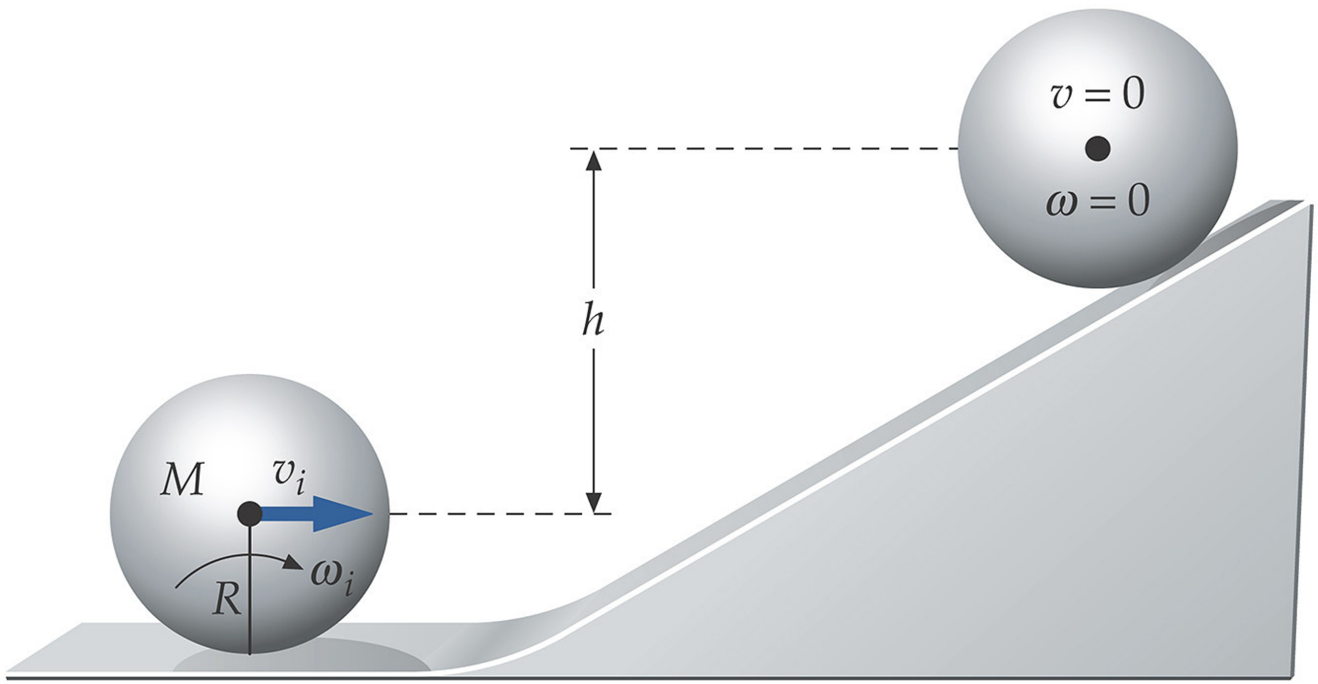
$$T_1 = \frac{m_2}{m_1 + m_2} m_1 g$$

$$T_2 = \frac{m_1}{m_1 + m_2} m_2 g$$

$$T_1 = T_2$$

A Bowling Ball

Only KE and GPE in this problem



A Bowling Ball

PE at the top = KE at the bottom

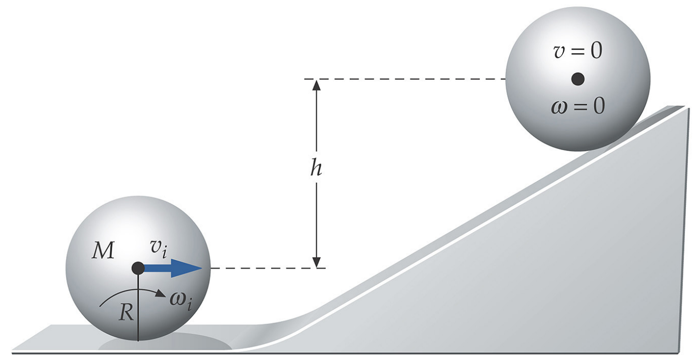
$$Mgh = \tfrac{1}{2} Mv_{cm}^2 + \tfrac{1}{2} I_{cm}\, \omega_i^2$$

$$Mgh = \tfrac{1}{2} Mv_{cm}^2 + \tfrac{1}{2} \beta\, MR^2 \frac{v_{cm}^2}{R^2}$$

$$Mgh = \tfrac{1}{2} Mv_{cm}^2 + \tfrac{1}{2} \beta\, Mv_{cm}^2$$

$$2gh = v_{cm}^2 (1 + \beta)$$

$$v_{cm} = \sqrt{\frac{2gh}{1+\beta}}$$

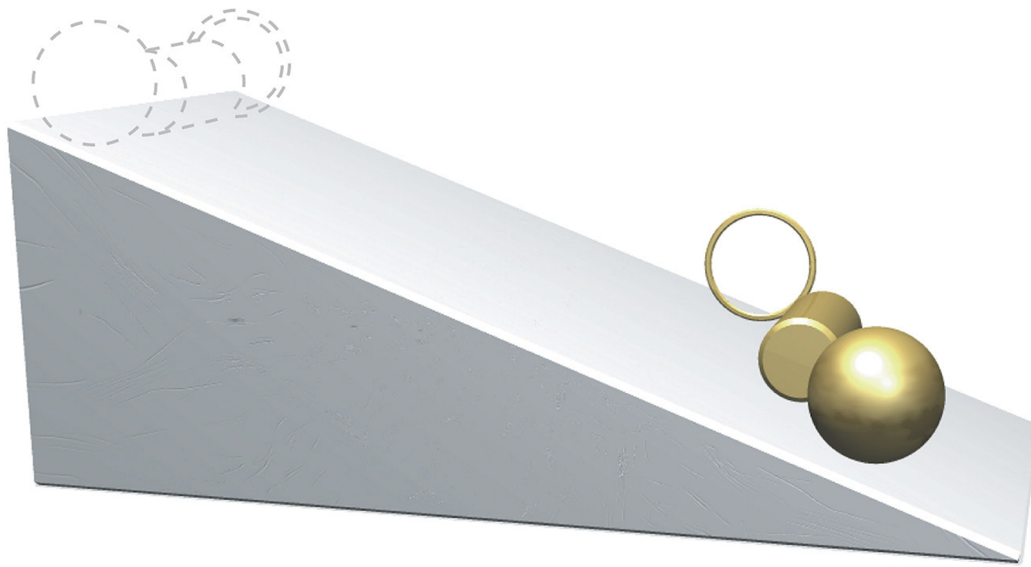


β-Values

Sphere - Hollow = 2/3

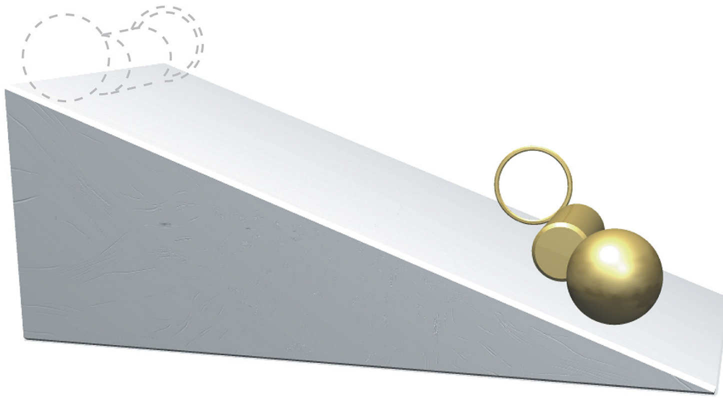
Sphere - Solid = 2/5

Loop vs Disk vs Sphere



This is the classic race between similarly shaped objects of different mass distributions. We might want to add a frictionless block for comparison.

Loop vs Disk vs Sphere



Loop: $\beta = 1$

Disk: $\beta = 1/2$

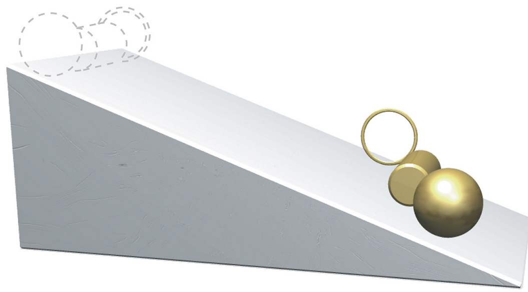
Sphere: $\beta = 2/5$

Perpendicular length = Radius

All moment of inertia formulas have a similar form. They are linear in the mass, quadratic in a characteristic length perpendicular to the axis of rotation and have a dimensionless factor β.

$$I = \beta M L^2$$

Loop vs Disk vs Sphere

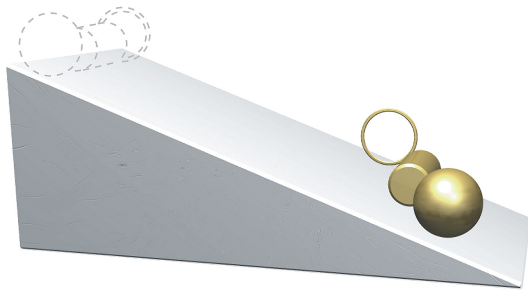


To determine the winner of the race we want to calculate the linear velocity of the cm of each of the objects.

For the frictionless block we need only consider the vertical drop of its center of mass and set the change in GPE equal to the translational KE of its CM.

$$v_{CM} = \sqrt{2gh}$$

Loop vs Disk vs Sphere vs Block



The block wins!

Loop: $\beta = 1$

Disk: $\beta = 1/2$

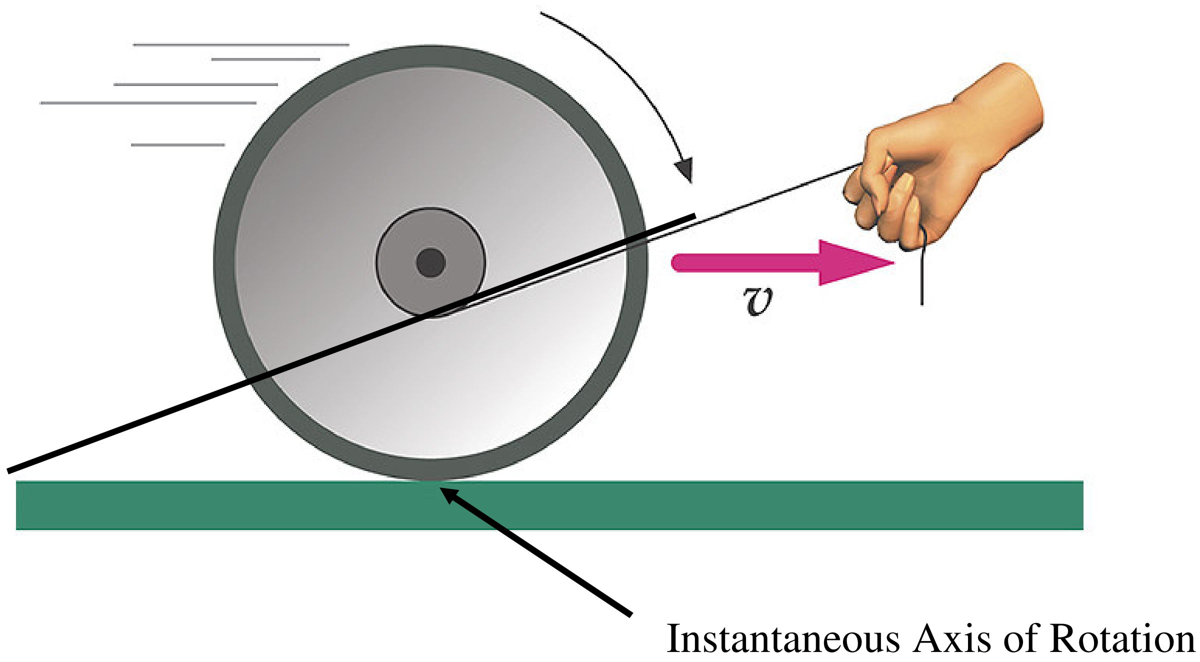
Sphere: $\beta = 2/5$

Block: $\beta = 0$

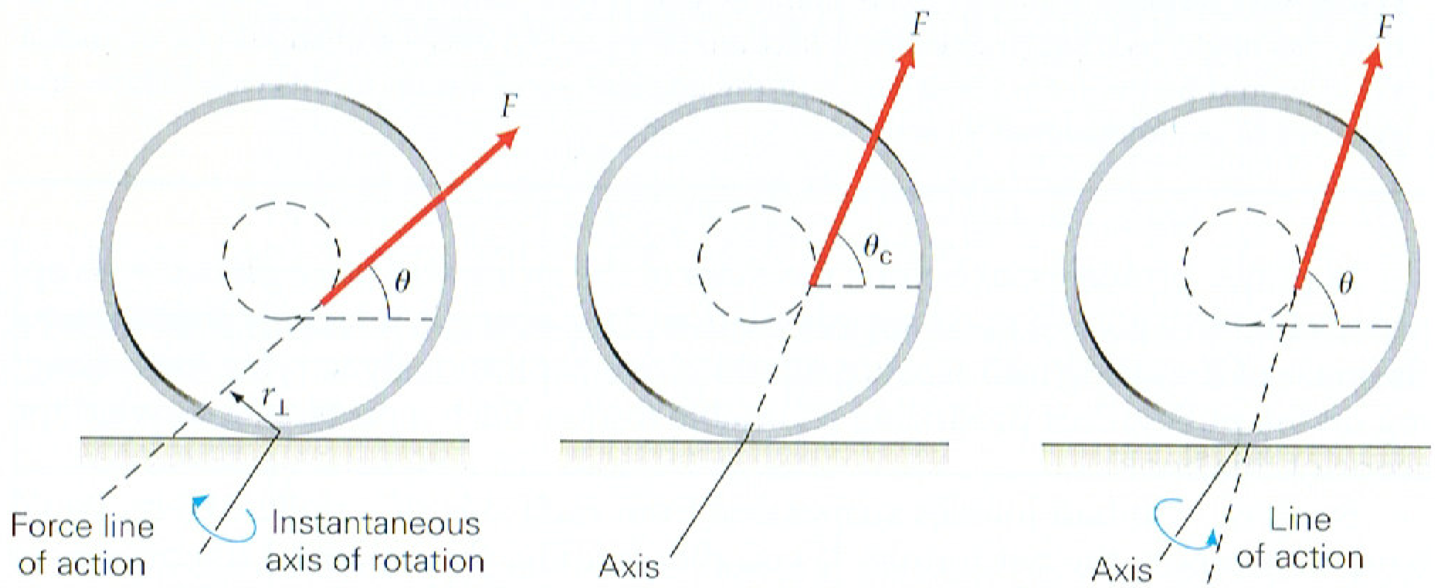
For the other three objects we also need to consider the vertical drop of its center of mass but we set the change in GPE equal to the translational KE of its CM PLUS the rotational KE.

$$v_{CM} = \sqrt{\frac{2gh}{1+\beta}}$$

Instantaneous Axis of Rotation



Instantaneous Axis of Rotation



(a) Rolls to right

(b) $\theta = \theta_c$, in rotational equilibrium does not roll

(c) $\theta > \theta_c$, rolls to left ($r_\perp$ to the right)

Rotational and Linear Analogs

Rotational Motion		Linear Motion	
Angular displacement	$\Delta\theta$	Displacement	Δx
Angular velocity	$\omega = \dfrac{d\theta}{dt}$	Velocity	$v_x = \dfrac{dx}{dt}$
Angular acceleration	$\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$	Acceleration	$a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}$
Constant-angular-acceleration equations	$\omega = \omega_0 + \alpha t$	Constant-acceleration equations	$v_x = v_{0x} + a_x t$
	$\Delta\theta = \omega_{\text{av}}\,\Delta t$		$\Delta x = v_{\text{av}\,x}\,\Delta t$
	$\omega_{\text{av}} = \frac{1}{2}(\omega_0 + \omega)$		$v_{\text{av}\,x} = \frac{1}{2}(v_{0x} + v_x)$
	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$		$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$
	$\omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta$		$v_x^2 = v_{0x}^2 + 2a_x\,\Delta x$
Torque	τ	Force	F_x
Moment of inertia	I	Mass	m
Work	$dW = \tau\,d\theta$	Work	$dW = F_x\,dx$
Kinetic energy	$K = \frac{1}{2}I\omega^2$	Kinetic energy	$K = \frac{1}{2}mv^2$
Power	$P = \tau\omega$	Power	$P = F_x v_x$
Angular momentum*	$L = I\omega$	Momentum	$p_x = mv_x$
Newton's second law	$\tau_{\text{net}} = I\alpha = \dfrac{dL}{dt}$	Newton's second law	$F_{\text{net}x} = ma_x = \dfrac{dp_x}{dt}$

Extra Slides