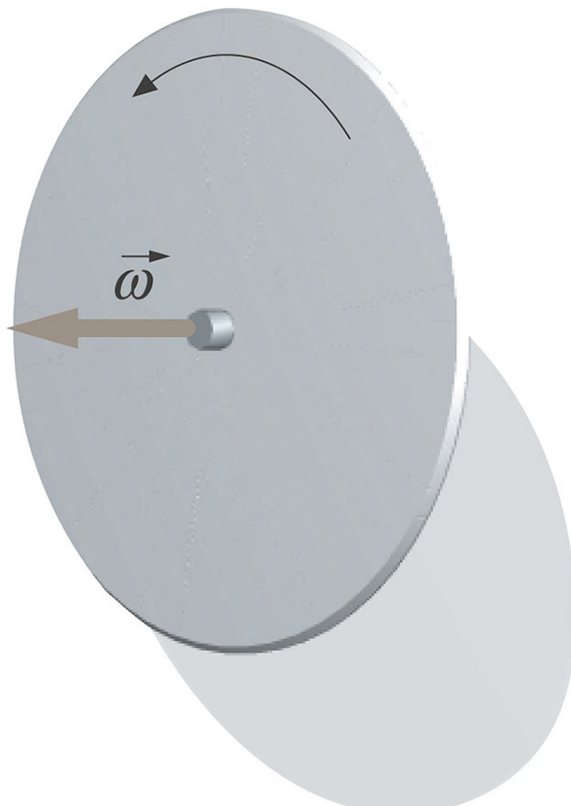
Chapter 11

Angular Momentum

Angular Momentum

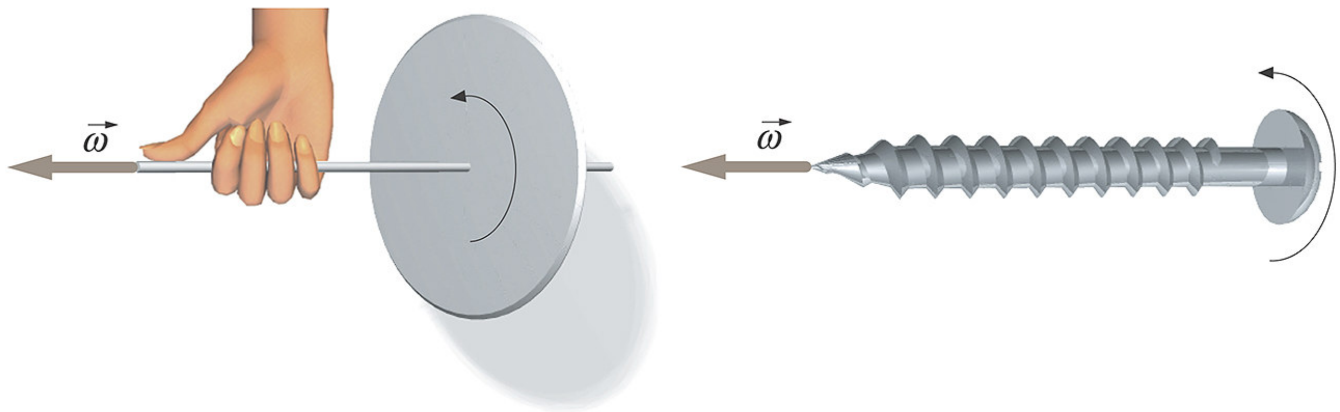
- The Vector Nature of Rotation
- Torque and Angular Momentum
- Conservation of Angular Momentum

The Vector Nature of Rotation



- All of our angular quantities θ, ω and α are vectors.
- Their directions are along the axis of rotation.
- Until now we had been dealing with just the scalar nature of these quantities and now we will study their vector characteristics.

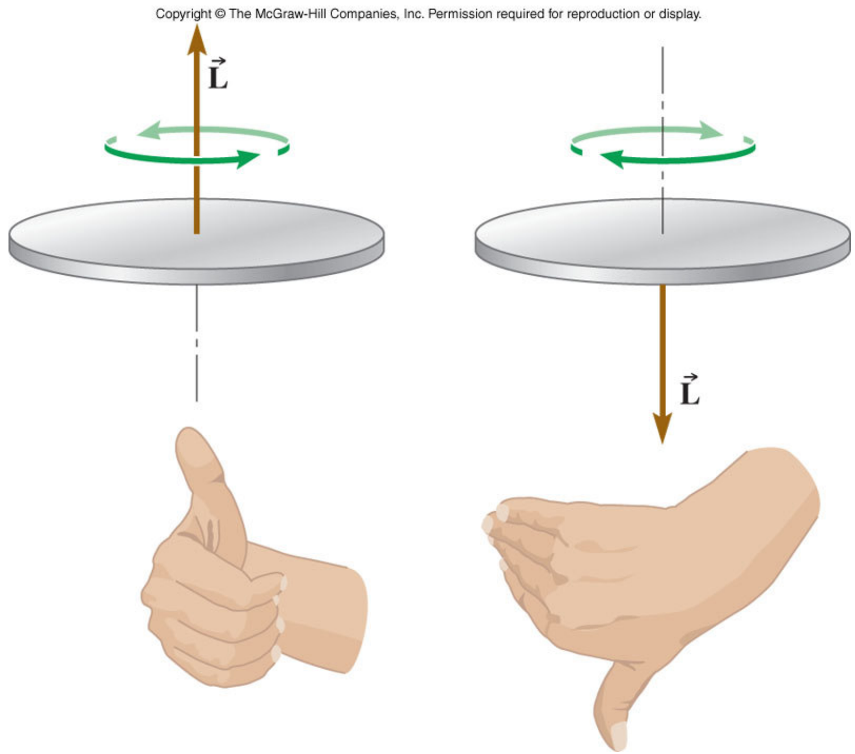
The Vector Nature of Rotation



In rotation the direction of the vectors can be determined by using the right-hand rule

The Right-Hand Rule

Curl the fingers of your right hand so that they curl in the direction a point on the object moves, and your thumb will point in the direction of the angular momentum.

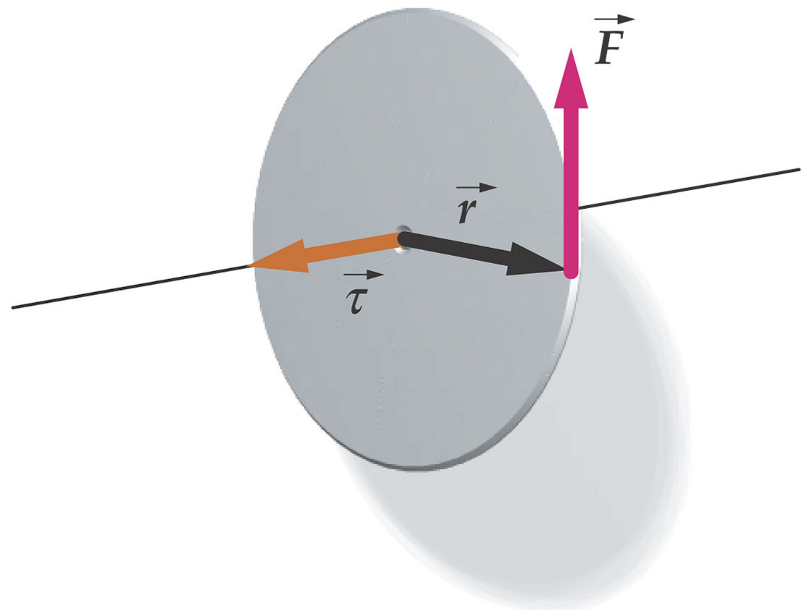


Angular Momentum is also an example of a vector cross product

The Vector Nature of Rotation

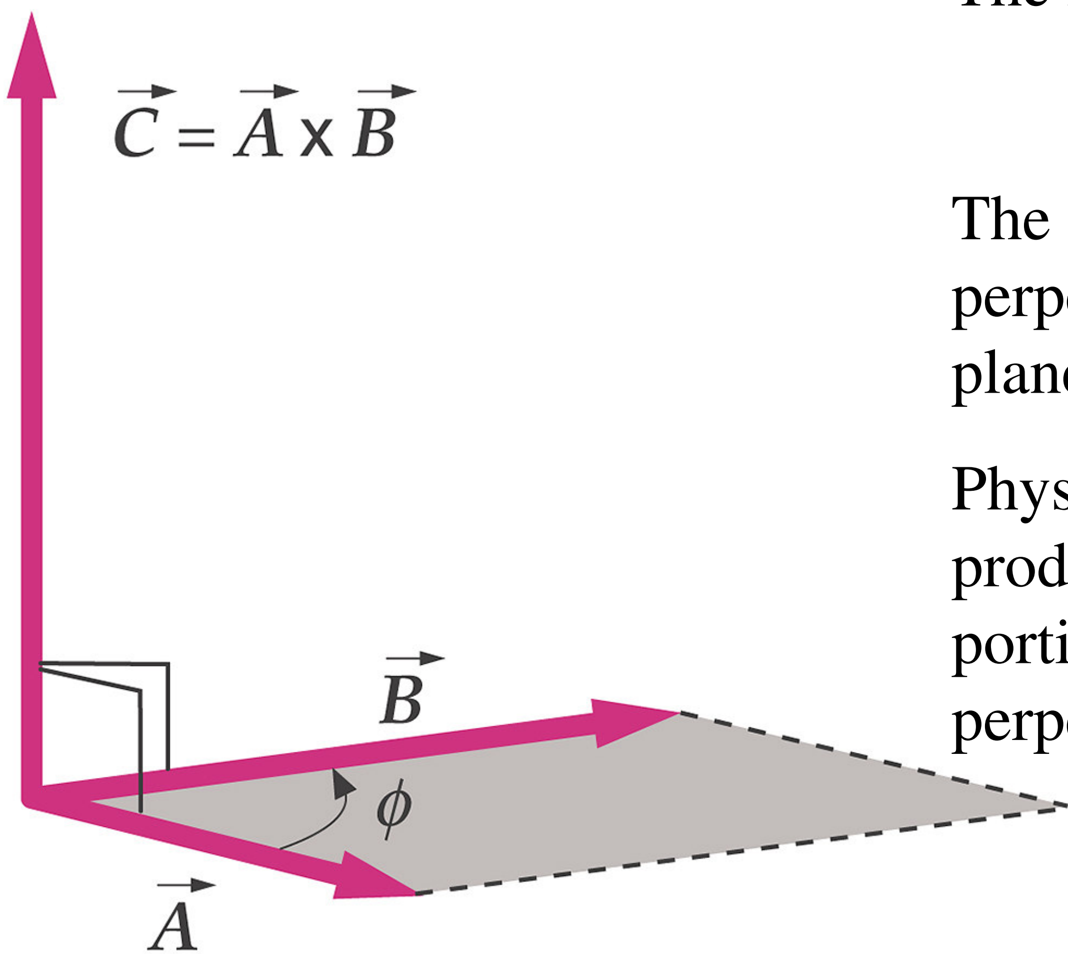
The figure is illustrating the creation of a torque (τ) (twisting force) on a disk by the application of a tangential force F.

It also obey a right-hand rule



The vector cross product is a mathematical statement of the right-hand rule.

The Vector Cross Product



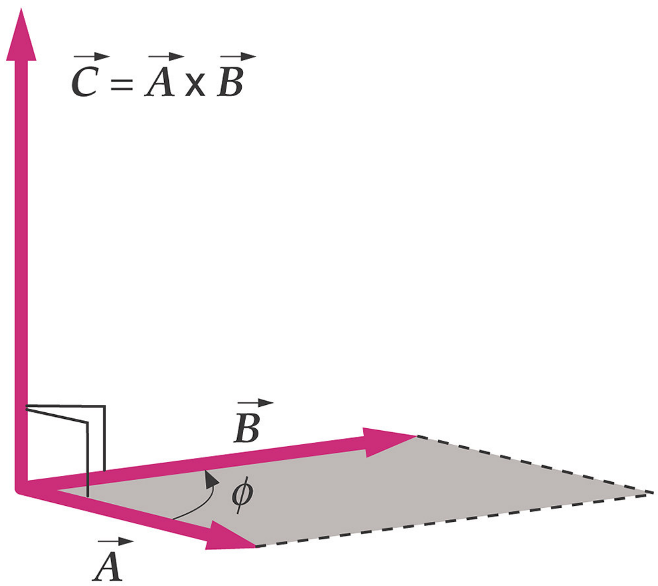
The magnitude of C

C = ABsin(Φ)

The direction of C is perpendicular to the plane of A and B.

Physically it means the product of A and the portion of B that is perpendicular to A.

The Cross Product by Components



$$A \times B = \text{vector}$$

$$(A \times B)_z = A_x B_y - A_y B_x$$

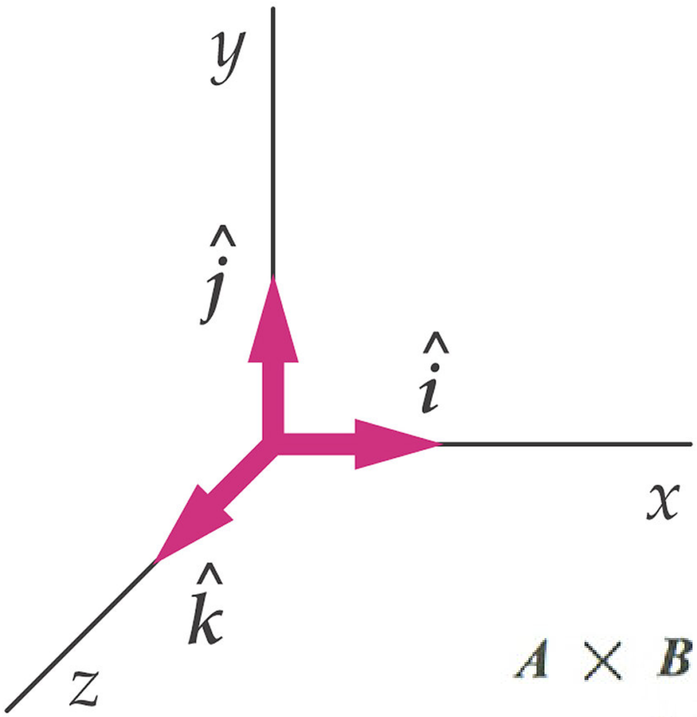
$$(A \times B)_x = A_y B_z - A_z B_y$$

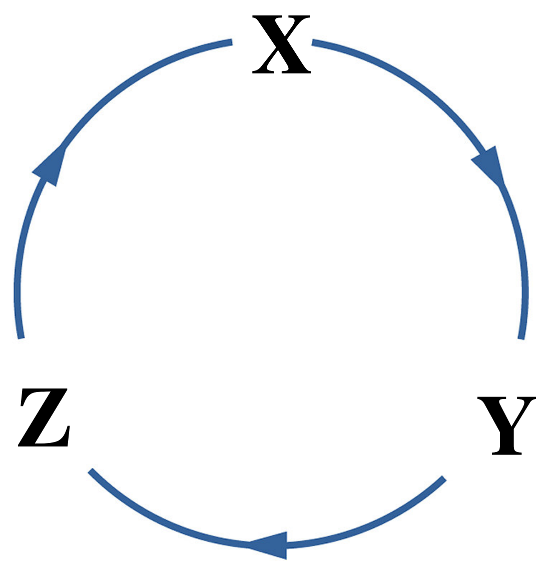
$$(A \times B)_y = A_z B_x - A_x B_z$$

Since A and B are in the x-y plane A x B is along the z-axis.

$$A \times B = (A \times B)_z = A_x B_y - A_y B_x$$

Memorizing the Cross Product





$$\boldsymbol{A} \times \boldsymbol{B} = \text{vector}$$

$$(\boldsymbol{A} \times \boldsymbol{B})_z = A_x B_y - A_y B_x$$

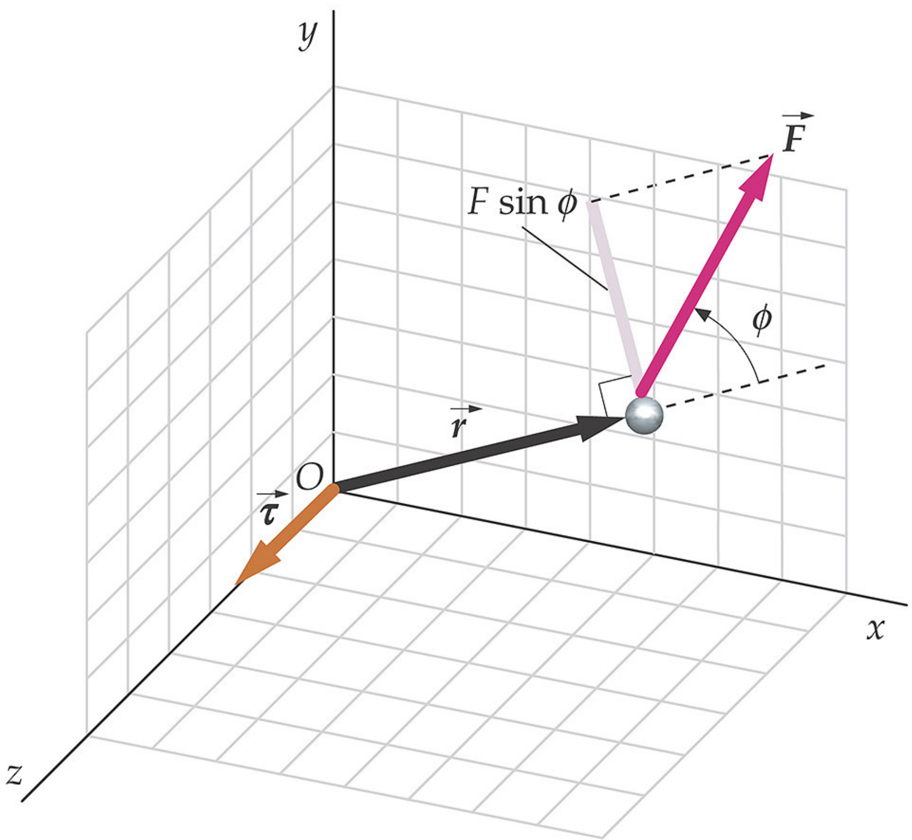
$$(\boldsymbol{A} \times \boldsymbol{B})_x = A_y B_z - A_z B_y$$

$$(\boldsymbol{A} \times \boldsymbol{B})_y = A_z B_x - A_x B_z$$

Torque and Angular Momentum

The Vector Nature of Torque

With torque we move from the plane to 3 dimensions

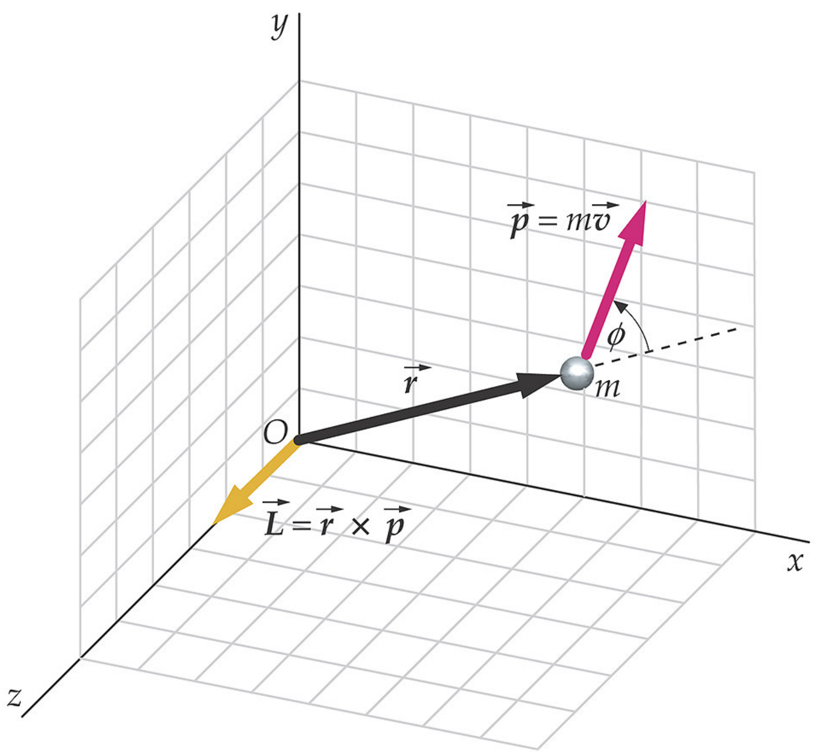


The vectors $\vec{r}$ and $\vec{F}$ are in the x-y plane. The torque is the result of a cross product operation. This will result in the torque vector pointing in the z-direction, perpendicular to the x-y plane.

The Vector Nature of Angular Momentum

Angular momentum is a vector. Its direction is defined with a **right-hand rule**.

The Vector Nature of Angular Momentum

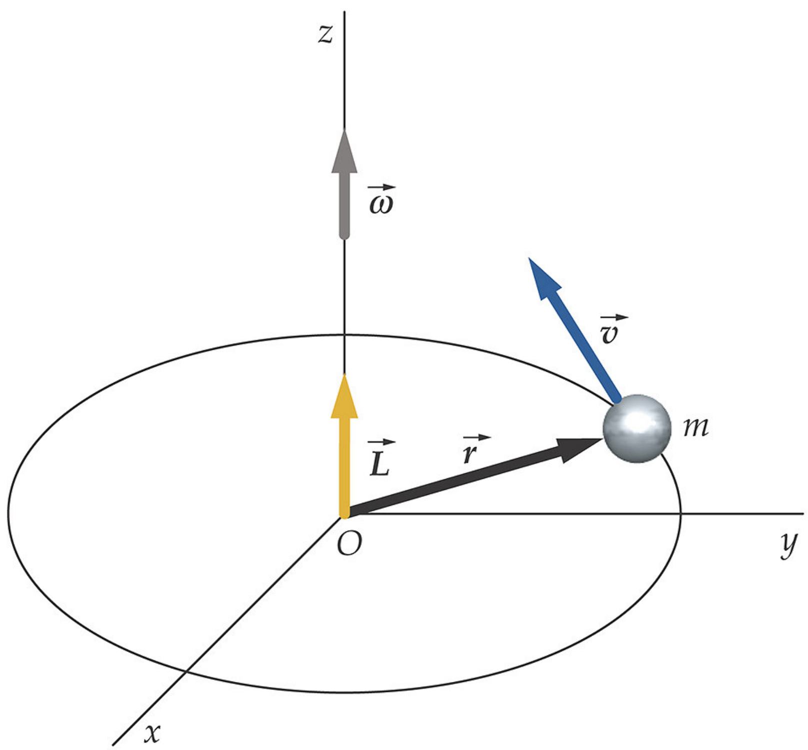


The vectors $\vec{r}$ and $\vec{p}$ are in the x-y plane. The angular momentum is the result of a cross product operation. This will result in an angular momentum vector pointing in the z-direction, perpendicular to the x-y plane.

Motion in a Plane

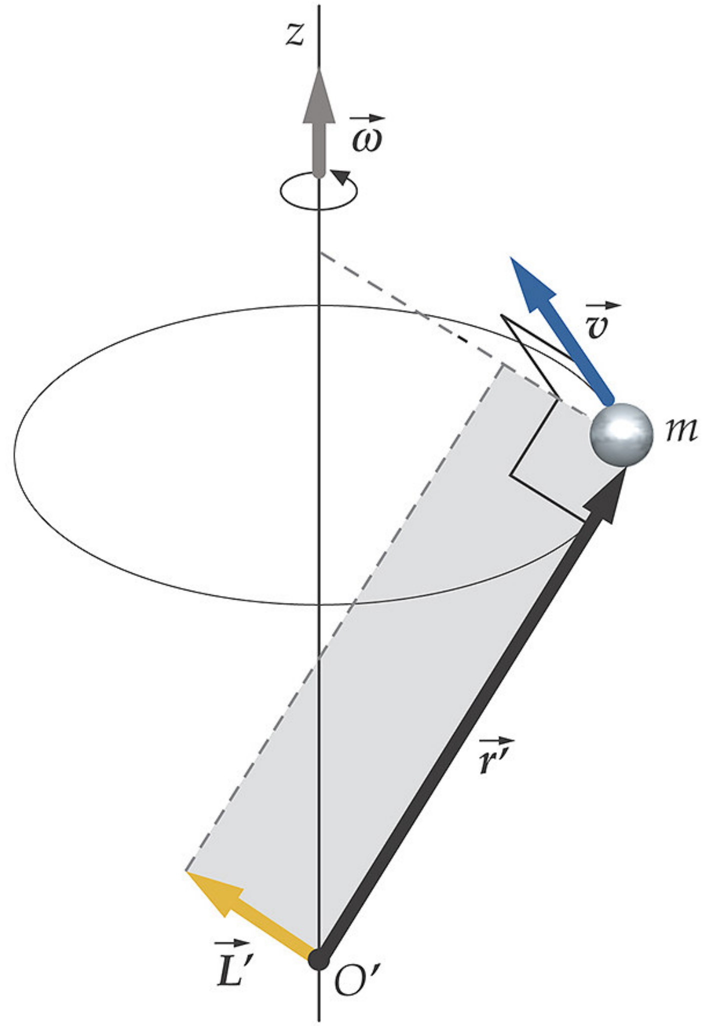
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = rmv\hat{k} = mr^2\omega\hat{k} = m\,r^2\,\vec{\omega} = I\vec{\omega}$$

$$\vec{L} = mr^2\,\vec{\omega} = I\vec{\omega}$$

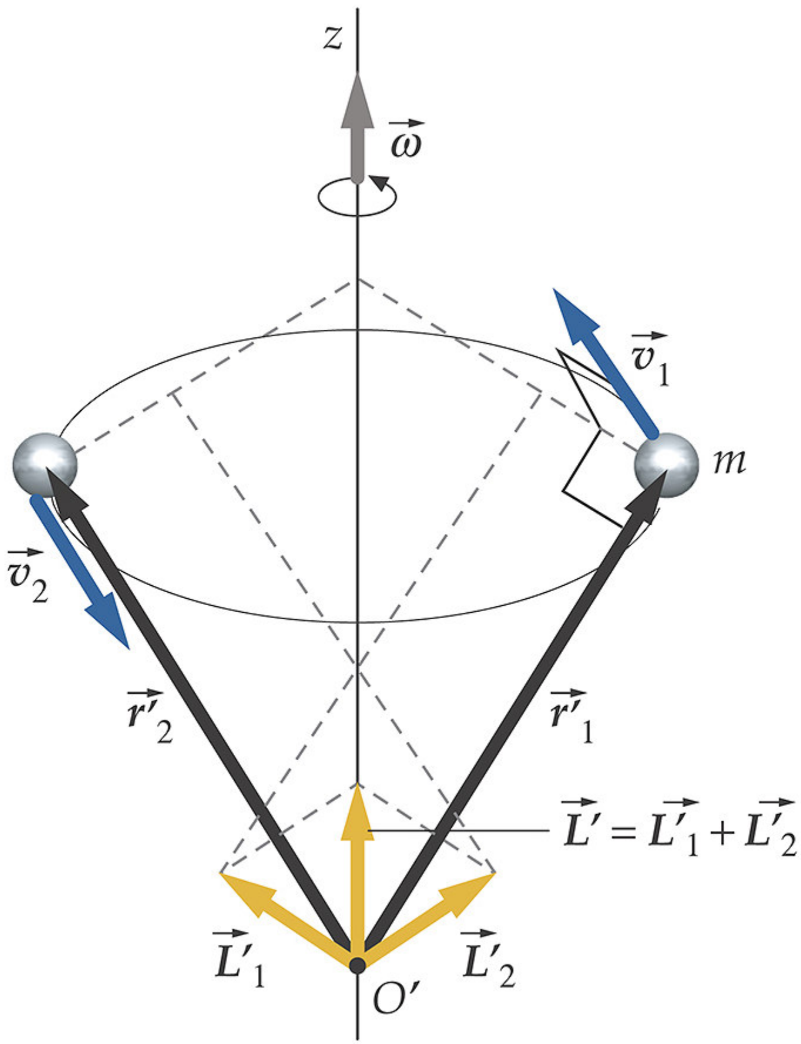


Angular Momentum & Symmetry

Without symmetry about the axis of rotation $\vec{\omega}$ and $\vec{L}$ are not co-axial.



Angular Momentum & Symmetry



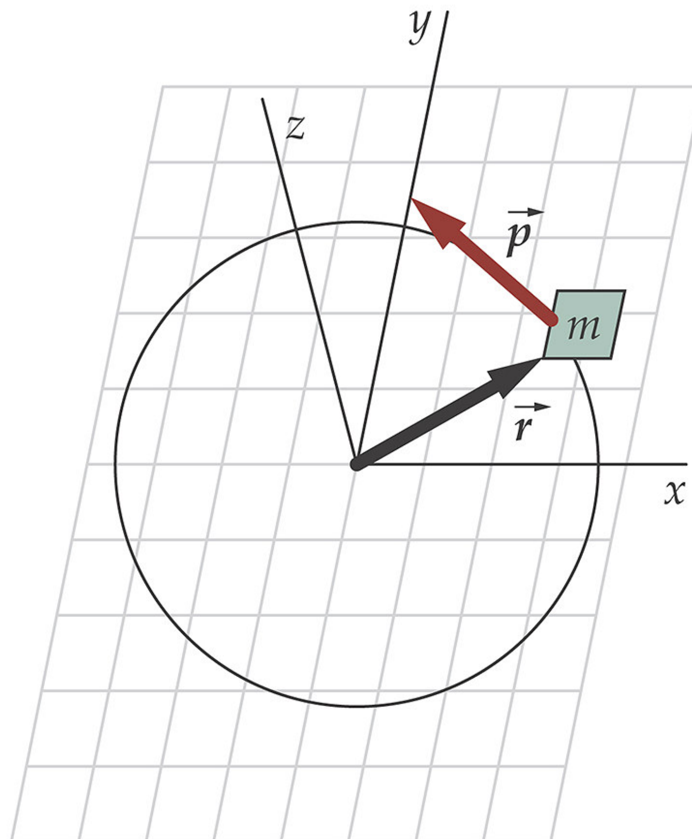
With symmetry about the axis of rotation $\vec{\omega}$ and $\vec{L}$ are co-axial.

This is the typical symmetry that we will be studying.

Angular Momentum About an Origin

Consider this a point mass.

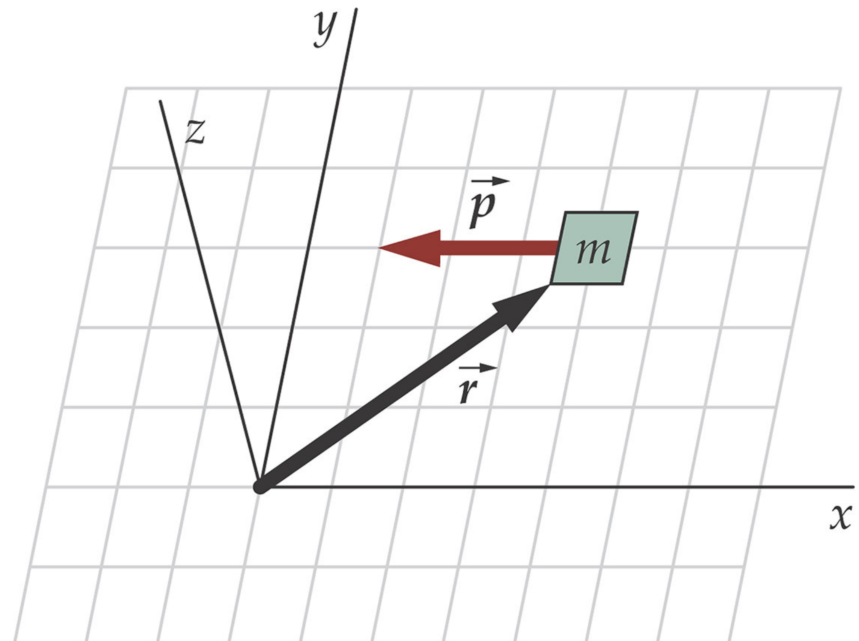
$$\vec{L} = \vec{r} \times \vec{p} = rmv\sin(90^0)\hat{k} = mrv\hat{k}$$



Angular Momentum About an Origin

$$\vec{L} = \vec{r} \times \vec{p} = \left(x\hat{i} + y_0\hat{j}\right) \times \left(-mv\hat{i}\right)$$

$$\vec{L} = -y_0 mv\left(\hat{j} \times \hat{i}\right) = y_0 mv\hat{k}$$

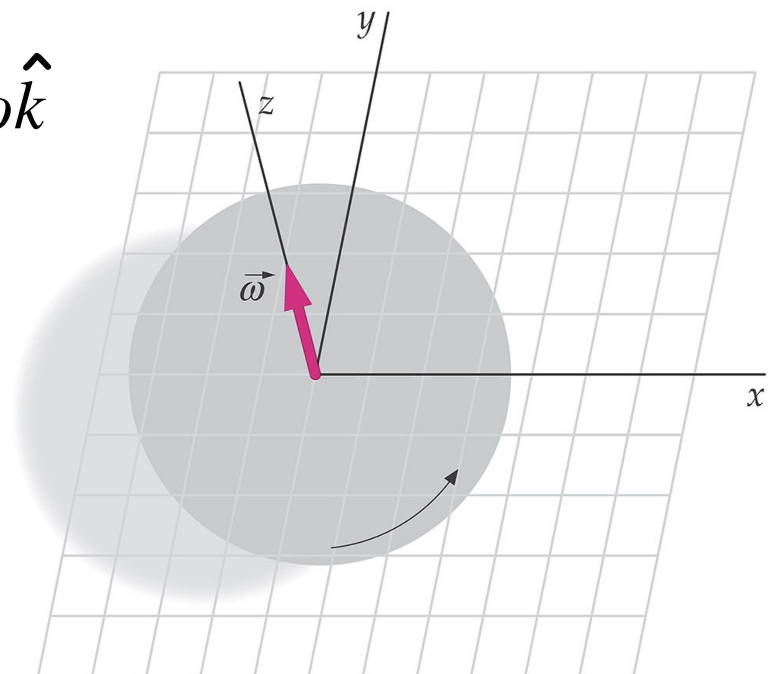


Consider this a point mass.

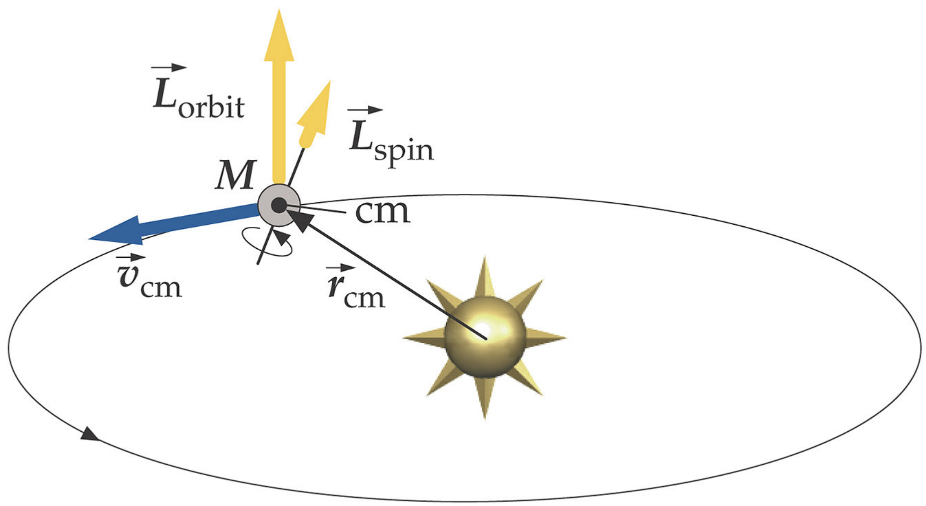
Angular Momentum About an Origin

Uniform mass

$$\vec{L} = I\vec{\omega} = I\omega\hat{k} = \tfrac{1}{2}MR^2\omega\hat{k}$$



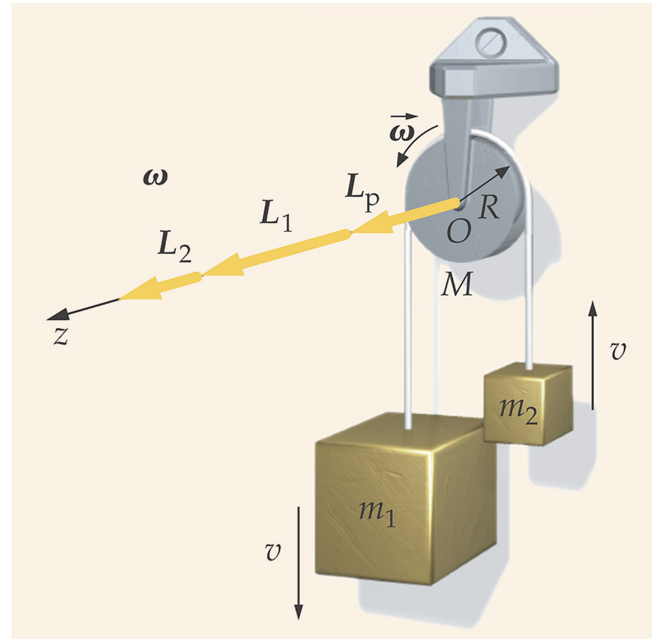
Orbital and Spin Angular Momentum



This decomposition of angular momentum into orbital and spin components is also useful in atomic physics and quantum mechanics

Atwood - Once More

- Massive pulley with friction between the pulley and string - no slipping.
- Pulley has a frictionless bearing.
- Massless string.



The first approach will be to set the net torque on the pulley equal to the rate of change of the angular momentum

$$\sum \tau_{ext\ z} = \frac{dL_z}{dt}$$

The equation will be solved for α and related to the tangential acceleration, "a", by the non-slip condition.

Atwood - Once More

$$\sum \tau_{ext\ z} = \frac{dL_z}{dt}$$

Before writing a general equation we identify those forces that will create a torque.

$$F_n = N$$

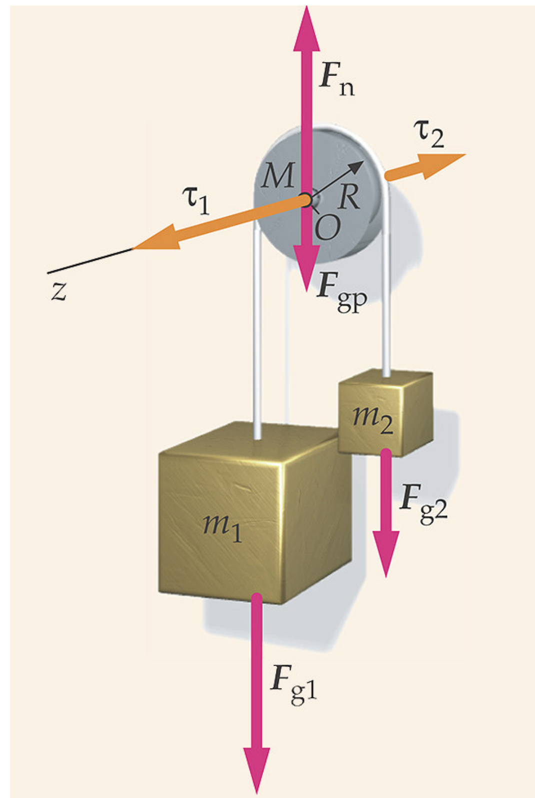
$$F_{gp} = Mg$$

$$F_{g1} = m_1 g$$

$$F_{g2} = m_2 g$$

We need to re-label the variables so that we recognize our old friends.

Only m_1 and m_2 can create torques. The others pass through the axis of rotation.



$$\sum \tau_{ext\ z} = \tau_1 + \tau_2$$

The Atwood Pulley - Close Up

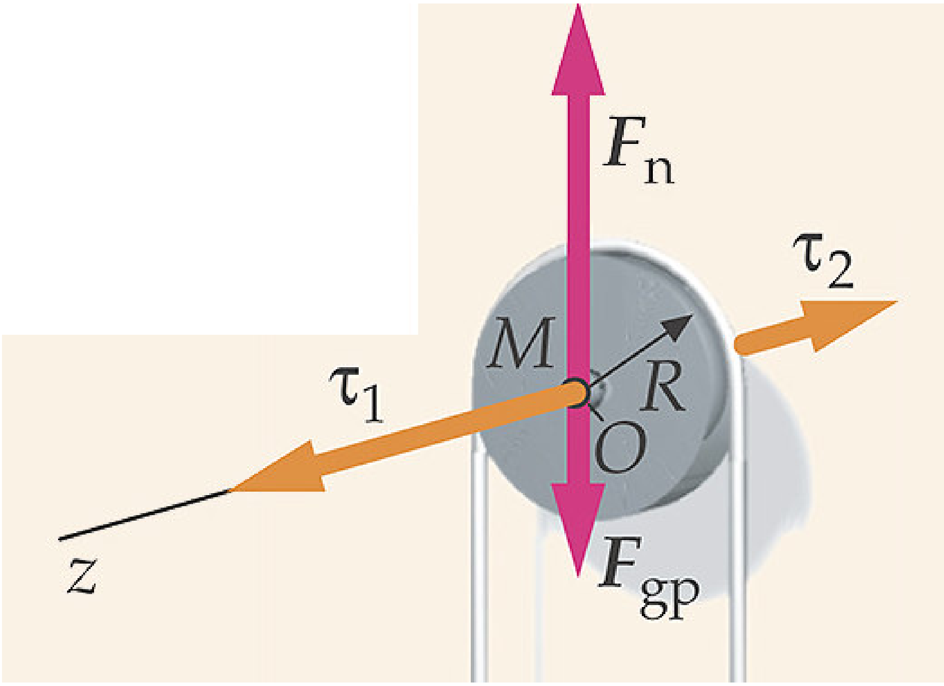
Massive pulley with friction between the pulley and string.

$$\sum \tau_{ext\ z} = \tau_1 + \tau_2$$

$$\sum \tau_{ext\ z} = m_1 gR - m_2 gR$$

$$L_z = L_1 + L_2 + L_p$$

$$L_z = m_1 vR + m_2 vR + I\omega$$



The Atwood Pulley - Close Up

$$\sum \tau_{ext\ z} = \frac{dL_z}{dt}$$

$$\sum \tau_{ext\ z} = m_1 gR - m_2 gR = \frac{d}{dt}\left(m_1 vR + m_2 vR + I\omega\right)$$

$$m_1 gR - m_2 gR = \left(m_1 + m_2\right) Ra + I\alpha$$

$$m_1 gR - m_2 gR = \left(m_1 + m_2\right) Ra + \tfrac{1}{2} MR^2 \frac{a}{R}$$

Solve the above equation for the acceleration a.

The Atwood Pulley - Close Up

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}\, g$$

$$\alpha = \frac{a}{R} = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}\,\frac{g}{R}$$

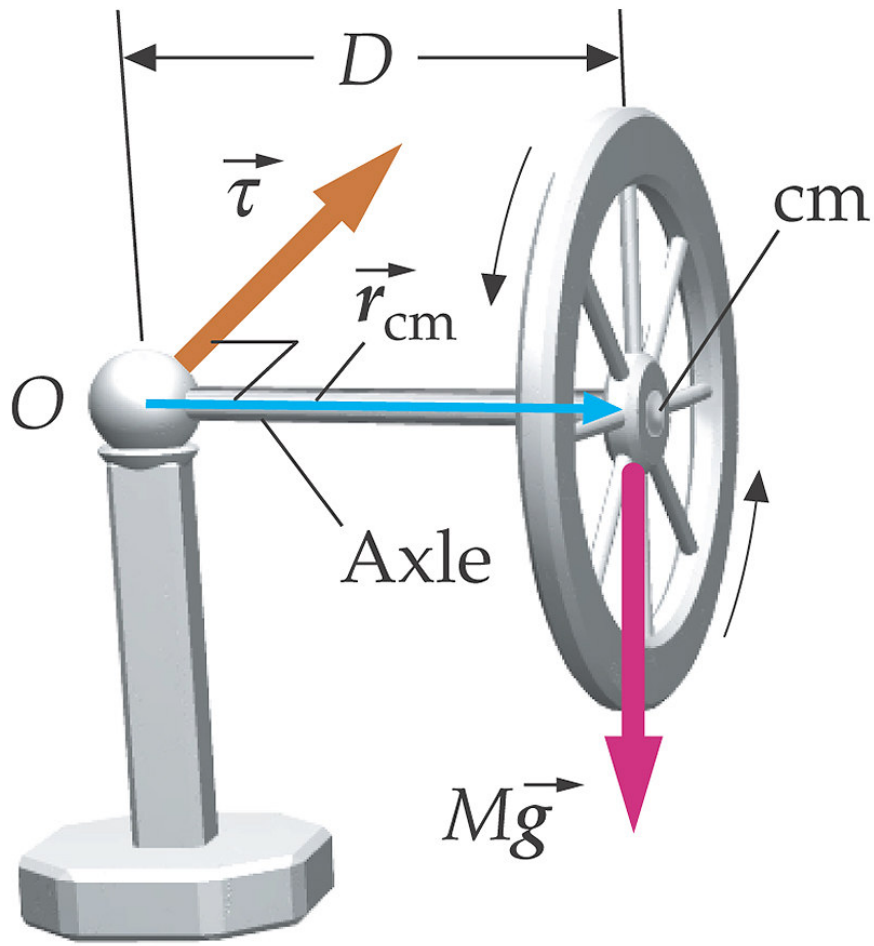
Conservation of Angular Momentum

If $\sum \vec{\tau}_{ext} = \dfrac{d\vec{L}}{dt} = 0$

$Then\ \vec{L}\ is\ constant$

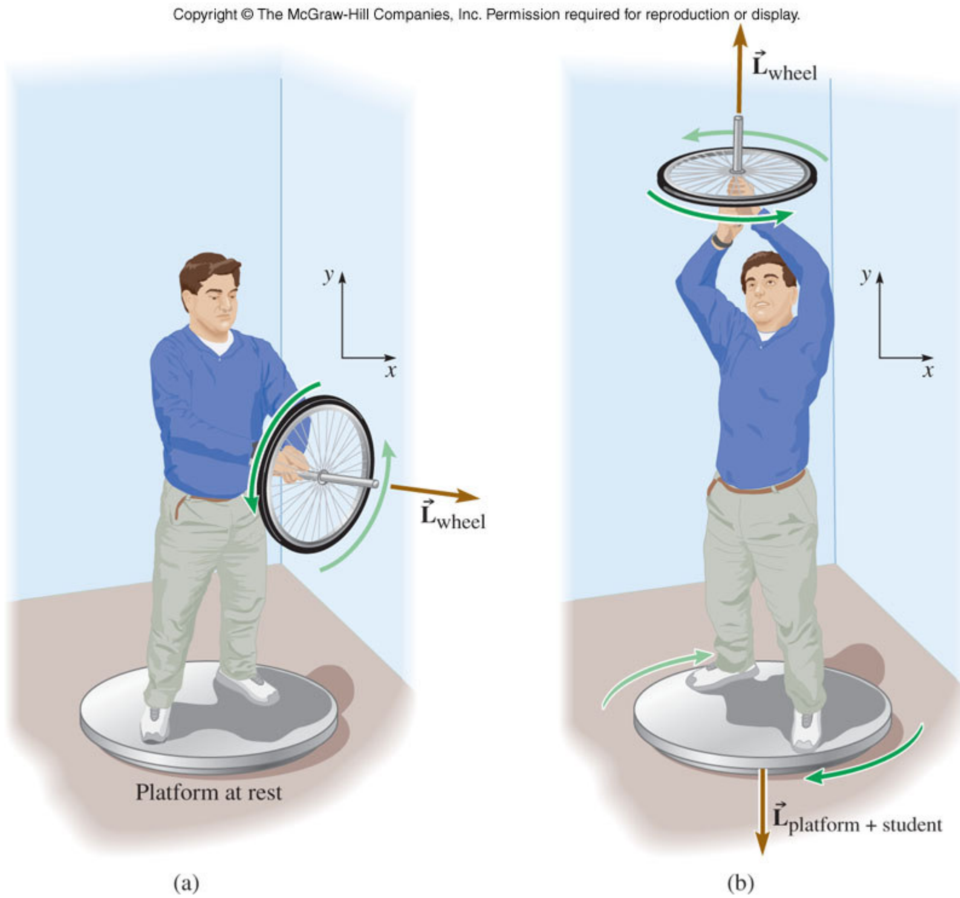
In both magnitude and direction

The Gyroscope Geometry



Angular Momentum Demo

Consider a person holding a spinning wheel. When viewed from the front, the wheel spins CCW.



Holding the wheel horizontal, they step on to a platform that is free to rotate about a vertical axis.

Angular Momentum Demo

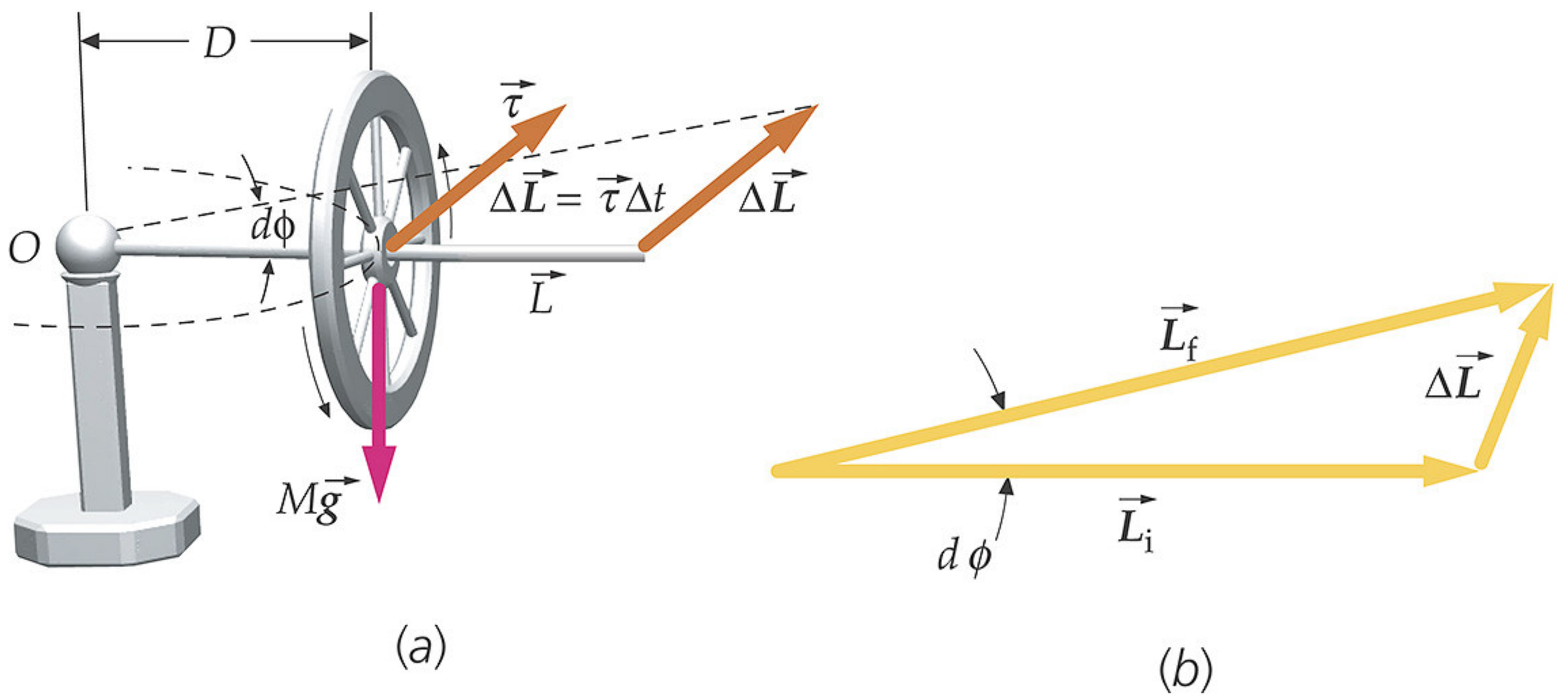
Initially there is no angular momentum about the vertical axis. When the wheel is moved so that it has angular momentum about this axis, the platform must spin in the opposite direction so that the net angular momentum stays zero.

This is a result of conserving angular momentum.

Is angular momentum conserved about the direction of the wheel's initial, horizontal axis?

No, that angular momentum is not conserved.

The earth exerts a torque on the system (platform + person), thus angular momentum is not conserved here.

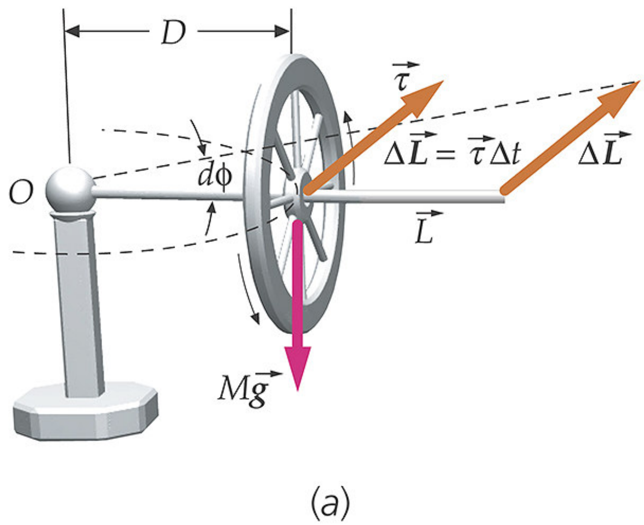


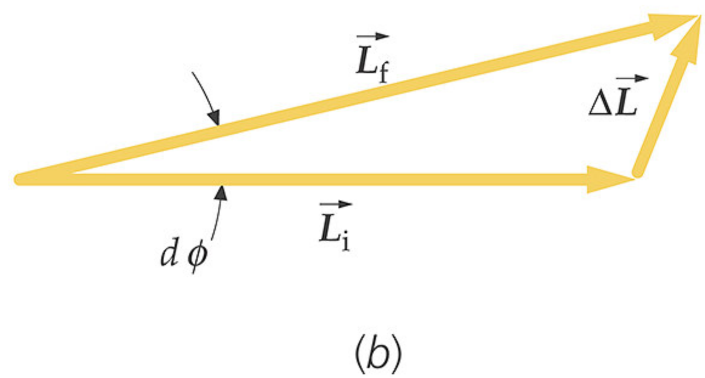
What is ω_p ,the angular velocity of the precession?

$$\tau = \frac{dL}{dt} \quad dL = \tau dt \quad \tau = DMg$$

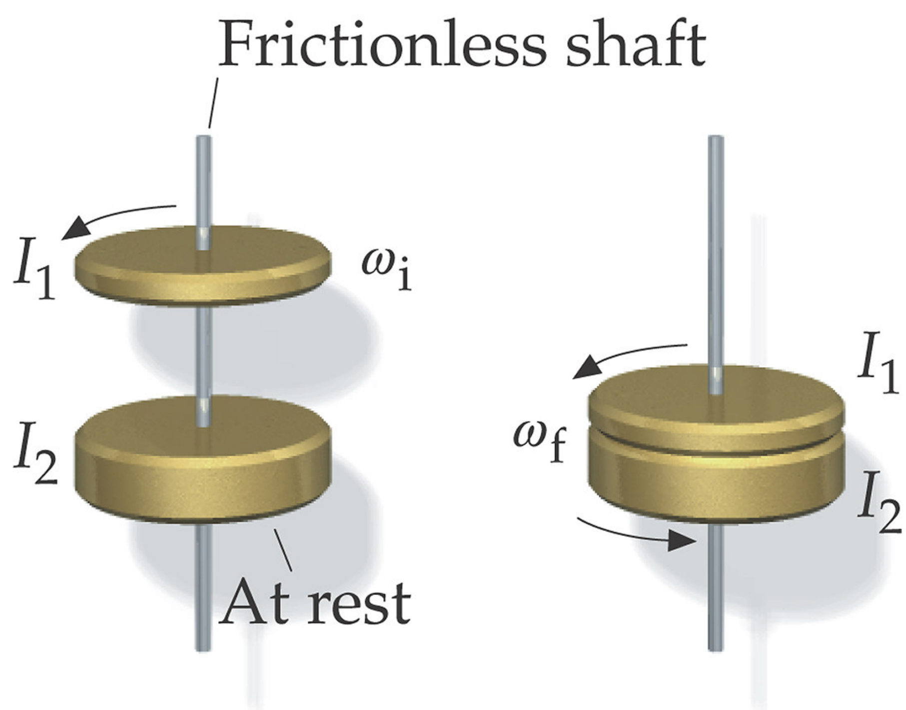
$$d\varphi = \frac{dL}{L} = \frac{DMgdt}{L}$$

$$\omega_p = \frac{d\varphi}{dt} = \frac{MgD}{L}$$





Conservation of Angular Momentum



Extra Slides