

Mechanics - Basic Physical Concepts

Math: Circle: $2\pi r$, πr^2 ; Sphere: $4\pi r^2$, $(4/3)\pi r^3$
Quadratic Eq.: $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Cartesian and polar coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}$$

Trigonometry: $\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta)$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}, \quad 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

Vector algebra: $\vec{A} = (A_x, A_y) = A_x \hat{i} + A_y \hat{j}$

Resultant: $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y)$

Dot: $\vec{A} \cdot \vec{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z$

Cross product: $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$C = AB \sin \theta = A_{\perp} B = AB_{\perp}$, use right hand rule

Calculus: $\frac{d}{dx} x^n = n x^{n-1}$, $\frac{d}{dx} \ln x = \frac{1}{x}$,

$$\frac{d}{d\theta} \sin \theta = \cos \theta, \quad \frac{d}{d\theta} \cos \theta = -\sin \theta, \quad \frac{d}{dx} \text{const} = 0$$

Measurements

Dimensional analysis: e.g.,

$$F = ma \rightarrow [M][L][T]^{-2}, \quad \text{or } F = m \frac{v^2}{r} \rightarrow [M][L][T]^{-2}$$

Summation: $\sum_{i=1}^N (a x_i + b) = a \sum_{i=1}^N x_i + b N$

Motion

One dimensional motion: $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$

Average values: $\bar{v} = \frac{s_f - s_i}{t_f - t_i}$, $\bar{a} = \frac{v_f - v_i}{t_f - t_i}$

One dimensional motion (constant acceleration):

$$v(t): v = v_0 + at$$

$$s(t): s = \bar{v}t = v_0 t + \frac{1}{2} at^2, \quad \bar{v} = \frac{v_0 + v}{2}$$

$$v(s): v^2 = v_0^2 + 2as$$

Nonuniform acceleration: $x = x_0 + v_0 t + \frac{1}{2} at^2 + \frac{1}{6} j t^3 + \frac{1}{24} s t^4 + \frac{1}{120} k t^5 + \frac{1}{720} p t^6 + \dots$ (jerk, snap, ...)

Projectile motion: $t_{rise} = t_{fall} = \frac{t_{trip}}{2} = \frac{v_{0y}}{g}$

$$h = \frac{1}{2} g t_{fall}^2, \quad R = v_{0x} t_{trip}$$

Circular: $a_c = \frac{v^2}{r}$, $v = \frac{2\pi r}{T}$, $f = \frac{1}{T}$ (Hertz=s⁻¹)

Curvilinear motion: $a = \sqrt{a_t^2 + a_r^2}$

Relative velocity: $\vec{v} = \vec{v}' + \vec{u}$

Law of Motion and applications

Force: $\vec{F} = m\vec{a}$, $F_g = mg$, $\vec{F}_{12} = -\vec{F}_{21}$

Circular motion: $a_c = \frac{v^2}{r}$, $v = \frac{2\pi r}{T} = 2\pi r f$

Friction: $F_{static} \leq \mu_s N$, $F_{kinetic} = \mu_k N$

Equilibrium (concurrent forces): $\sum_i \vec{F}_i = 0$

Energy

Work (for all F): $\Delta W = W_{A \rightarrow B} = W_B - W_A =$

$F_{\parallel} s = F s \cos \theta = \vec{F} \cdot \vec{s} \rightarrow \int_A^B \vec{F} \cdot d\vec{s}$ (in Joules)

Effects due to work done: $F_{ext} = ma + F_c + f_{nc}$

$W_{ext}|_{A \rightarrow B} = K_B - K_A + U_B - U_A + W_{diss}|_{A \rightarrow B}$

Kinetic energy: $K_B - K_A = \int_A^B m \vec{a} \cdot d\vec{s}$, $K = \frac{1}{2} m v^2$

K (conservative \vec{F}): $U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{s}$

$$U_{gravity} = mgy, \quad U_{spring} = \frac{1}{2} k x^2$$

From U to \vec{F} : $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

$$F_{gravity} = -\frac{\partial U}{\partial y} = -mg, \quad F_{spring} = -\frac{\partial U}{\partial x} = -kx$$

Equilibrium: $\frac{\partial U}{\partial x} = 0$, $\frac{\partial^2 U}{\partial x^2} > 0$ stable, < 0 unstable

Power: $P = \frac{dW}{dt} = F v_{\parallel} = F v \cos \theta = \vec{F} \cdot \vec{v}$ (Watts)

Collision

Impulse: $\vec{I} = \Delta \vec{p} = \vec{p}_f - \vec{p}_i \rightarrow \int_{t_i}^{t_f} \vec{F} dt$

Momentum: $\vec{p} = m\vec{v}$

Two-body: $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$$p_{cm} \equiv M v_{cm} = p_1 + p_2 = m_1 v_1 + m_2 v_2$$

$$F_{cm} \equiv F_1 + F_2 = m_1 a_1 + m_2 a_2 = M a_{cm}$$

$$K_1 + K_2 = K_1^* + K_2^* + K_{cm}$$

Two-body collision: $\vec{p}_i = \vec{p}_f = (m_1 + m_2) \vec{v}_{cm}$

$$v_i^* = v_i - v_{cm}, \quad v_i' = v_i^* + v_{cm}$$

Elastic: $v_1 - v_2 = -(v_1' - v_2')$,

$$v_i^* = -v_i^*, \quad v_i' = 2v_{cm} - v_i$$

Many body center of mass: $\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\int \vec{r} dm}{\int m_i}$

Force on cm: $\vec{F}_{ext} = \frac{d\vec{p}}{dt} = M \vec{a}_{cm}$, $\vec{p} = \sum \vec{p}_i$

Rotation of Rigid-Body

Kinematics: $\theta = \frac{s}{r}$, $\omega = \frac{v}{r}$, $\alpha = \frac{a_t}{r}$

Moment of inertia: $I = \sum m_i r_i^2 = \int r^2 dm$

$$I_{disk} = \frac{1}{2} M R^2, \quad I_{ring} = \frac{1}{2} M (R_1^2 + R_2^2)$$

$$I_{rod} = \frac{1}{12} M \ell^2, \quad I_{rectangle} = \frac{1}{12} M (a^2 + b^2)$$

$$I_{sphere} = \frac{2}{5} M R^2, \quad I_{spherical\ shell} = \frac{2}{3} M R^2$$

$$I = M (\text{Radius of gyration})^2, \quad I = I_{cm} + M D^2$$

Kinetic energies: $K_{rot} = \frac{1}{2} I \omega^2$, $K = K_{rot} + K_{cm}$

Angular momentum: $L = r m v = r m \omega r = I \omega$

Torque: $\tau = \frac{dL}{dt} = m \frac{dv}{dt} r = F r = I \frac{d\omega}{dt} = I \alpha$

$W_{ext} = \Delta K + \Delta U + W_f$, $K = K_{rot} + \frac{1}{2} m v^2$, $P = \tau \omega$

Rolling, angular momentum and torque

Rolling: $K = \frac{1}{2} (I_c + M R^2) \omega^2 = \frac{1}{2} \left(\frac{I_c}{R^2} + M \right) v^2$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$, $L = r_{\perp} p = I \omega$

Torque: $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$, $\tau = r_{\perp} F = I \alpha$

Gyroscope: $\omega_p = \frac{d\phi}{dt} = \frac{1}{L} \frac{dL}{dt} = \frac{\tau}{L} = \frac{m g h}{I \omega}$

Static equilibrium

$$\sum \vec{F}_i = 0, \quad \text{about any point } \sum \vec{\tau}_i = 0$$

Subdivisions: $\vec{r}_{cm} = \frac{m_A \vec{r}_{Acm} + m_B \vec{r}_{Bcm}}{m_A + m_B}$

Elastic modulus = stress/strain

stress: F/A

strain: $\Delta L/L$, $\theta \approx \Delta x/h$, $-\Delta V/V$

Oscillation motion

$$f = \frac{1}{T}, \quad \omega = \frac{2\pi}{T}$$

SHM: $a = \frac{d^2x}{dt^2} = -\omega^2 x, \quad \alpha = \frac{d^2\theta}{dt^2} = -\omega^2 \theta$
 $x = x_{max} \cos(\omega t + \delta), \quad x_{max} = A$
 $v = -v_{max} \sin(\omega t + \delta), \quad v_{max} = \omega A$
 $a = -a_{max} \cos(\omega t + \delta) = -\omega^2 x, \quad a_{max} = \omega^2 A$
 $E = K + U = K_{max} = \frac{1}{2} m (\omega A)^2 = U_{max} = \frac{1}{2} k A^2$
 Spring: $ma = -kx$
 Simple pendulum: $ma_\theta = m\alpha\ell = -mg \sin\theta$
 Physical pendulum: $\tau = I\alpha = -mgd \sin\theta$
 Torsion pendulum: $\tau = I\alpha = -\kappa\theta$

Gravity

$$\vec{F}_{21} = -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12}, \quad \text{for } r \geq R, \quad g(r) = G \frac{M}{r^2}$$

$$G = 6.67259 \times 10^{-11} \text{ N m}^2/\text{kg}^2$$

$$R_{earth} = 6370 \text{ km}, \quad M_{earth} = 5.98 \times 10^{24} \text{ kg}$$

Circular orbit: $a_c = \frac{v^2}{r} = \omega^2 r = \left(\frac{2\pi}{T}\right)^2 r = g(r)$
 $U = -G \frac{mM}{r}, \quad E = U + K = -\frac{GMm}{2r}$
 $F = -\frac{dU}{dr} = -mG \frac{M}{r^2} = -m \frac{v^2}{r}$

Kepler's Laws of planetary motion:

i) elliptical orbit, $r = \frac{r_0}{1 - \epsilon \cos\theta}, \quad r_1 = \frac{r_0}{1 + \epsilon}, \quad r_2 = \frac{r_0}{1 - \epsilon}$
 ii) $L = r m \frac{\Delta r_\perp}{\Delta t} \rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2} r \frac{\Delta r_\perp}{\Delta t} = \frac{L}{2m} = \text{const.}$
 iii) $G \frac{M}{a^2} = \left(\frac{2\pi a}{T}\right)^2 \frac{1}{a}, \quad a = \frac{r_1 + r_2}{2}, \quad T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$
Escape kinetic energy: $E = K + U(R) = 0$

Fluid mechanics

Pascal: $P = \frac{F_{\perp 1}}{A_1} = \frac{F_{\perp 2}}{A_2}, \quad 1 \text{ atm} = 1.013 \times 10^5 \text{ N/m}^2$
Archimedes: $B = Mg, \quad \text{Pascal} = \text{N/m}^2$
 $P = P_{atm} + \rho gh, \quad \text{with } P = \frac{F_\perp}{A} \text{ and } \rho = \frac{m}{V}$
 $F = \int P dA \rightarrow \rho g \ell \int_0^h (h - y) dy$
Continuity equation: $Av = \text{constant}$
Bernoulli: $P + \frac{1}{2} \rho v^2 + \rho gy = \text{const}, \quad P \geq 0$

Wave motion

Traveling waves: $y = f(x - vt), \quad y = f(x + vt)$
In the positive x direction: $y = A \sin(kx - \omega t - \phi)$
 $T = \frac{1}{f}, \quad \omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}, \quad v = \frac{\omega}{k} = \frac{\lambda}{T}$

Along a string: $v = \sqrt{\frac{F}{\mu}}$

General: $\Delta E = \Delta K + \Delta U = \Delta K_{max}$

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2} \frac{\Delta m}{\Delta t} (\omega A)^2$$

Waves: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot \frac{\Delta x}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot v$

$$P = \frac{1}{2} \mu v (\omega A)^2, \quad \text{with } \mu = \frac{\Delta m}{\Delta x}$$

Circular: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta A} \cdot \frac{\Delta A}{\Delta r} \cdot \frac{\Delta r}{dt} = \frac{\Delta m}{\Delta A} \cdot 2\pi r v$

Spherical: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot 4\pi r^2 v$

Sound

$$v = \sqrt{\frac{B}{\rho}}, \quad s = s_{max} \cos(kx - \omega t - \phi)$$

$$\Delta P = -B \frac{\Delta V}{V} = -B \frac{\partial s}{\partial x}$$

$$\Delta P_{max} = B \kappa s_{max} = \rho v \omega s_{max}$$

Piston: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot \frac{\Delta \Delta x}{\Delta t} = \rho A v$

Intensity: $I = \frac{P}{A} = \frac{1}{2} \rho v (\omega s_{max})^2$

Intensity level: $\beta = 10 \log_{10} \frac{I}{I_0}, \quad I_0 = 10^{-12} \text{ W/m}^2$

Plane waves: $\psi(x, t) = c \sin(kx - \omega t)$

Circular waves: $\psi(r, t) = \frac{c}{\sqrt{r}} \sin(kr - \omega t)$

Spherical: $\psi(r, t) = \frac{c}{r} \sin(kr - \omega t)$

Doppler effect: $\lambda = vT, \quad f_0 = \frac{1}{T}, \quad f' = \frac{v'}{\lambda'}$

Here $v' = v_{sound} \pm v_{observer}$, is wave speed relative to moving observer and $\lambda' = (v_{sound} \pm v_{source})/f_0$, detected wave length established by moving source of frequency f_0 . $f_{received} = f_{reflected}$

Shock waves: Mach Number = $\frac{v_{source}}{v_{sound}} = \frac{1}{\sin\theta}$

Superposition of waves

Phase difference: $\sin(kx - \omega t) + \sin(kx - \omega t - \phi)$

Standing waves: $\sin(kx - \omega t) + \sin(kx + \omega t)$

Beats: $\sin(kx - \omega_1 t) + \sin(kx - \omega_2 t)$

Others: $a \cos(kx - \omega t) + b \sin(kx - \omega t)$

$y = \sin(kx - \omega t), \quad z = \sin(kx - \omega t)$

Fundamental modes: Sketch wave patterns

String: $\frac{\lambda}{2} = \ell, \quad \text{Rod clamped middle: } \frac{\lambda}{2} = \ell,$

Open-open pipe: $\frac{\lambda}{2} = \ell, \quad \text{Open-closed pipe: } \frac{\lambda}{4} = \ell$

Temperature and heat

Conversions: $F = \frac{9}{5} C + 32^\circ, \quad K = C + 273.15^\circ$

Constant volume gas thermometer: $T = aP + b$

Thermal expansion: $\alpha = \frac{1}{\ell} \frac{d\ell}{dT}, \quad \beta = \frac{1}{V} \frac{dV}{dT}$

$$\Delta \ell = \alpha \ell \Delta T, \quad \Delta A = 2\alpha A \Delta T, \quad \Delta V = 3\alpha V \Delta T$$

Ideal gas law: $PV = nRT = NkT$

$$R = 8.314510 \text{ J/mol/K} = 0.0821 \text{ L atm/mol/K}$$

$$k = 1.38 \times 10^{-23} \text{ J/K}, \quad N_A = 6.02 \times 10^{23}, \quad 1 \text{ cal} = 4.19 \text{ J}$$

Calorimetry: $\Delta Q = cm \Delta T, \quad \Delta Q = L \Delta m$

First law: $\Delta U = \Delta Q - \Delta W, \quad W = \int P dV$

Conduction: $H = \frac{\Delta Q}{\Delta t} = -kA \frac{\Delta T}{\Delta \ell}, \quad \Delta T_i = \frac{-H \ell_i}{k_i}$

Stefan's law: $P = \sigma A e T^4, \quad \sigma = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \text{K}^4}$

Kinetic theory of gas

Ideal gas: $\Delta p_x = 2m v_x, \quad F = \frac{\Delta p_x}{\Delta t} = \frac{m v_x^2}{d}$

Pressure: $P = \frac{N\bar{F}}{A} = \frac{mN}{V} \overline{v_x^2} = \frac{mN}{3V} \overline{v^2}$

$$P = \frac{2}{3} \frac{N}{V} \bar{K}, \quad \bar{K}_x = \frac{\bar{K}}{3} = \frac{1}{2} kT, \quad T = 273 + T_c,$$

$$PV = NkT, \quad n = N/N_A, \quad k = 1.38 \times 10^{-23} \text{ J/K},$$

$$N_A = 6.02214199 \times 10^{23} \text{ \#/kg/mole}$$

Constant V: $\Delta Q = \Delta U = n C_V \Delta T$

Constant P: $\Delta Q = n C_P \Delta T$

$$\gamma = \frac{C_P}{C_V}, \quad C_P - C_V = R$$

$$C_V = \frac{d}{2} R, \quad \text{for transl.+rot+vib, } d = 3 + 2 + 2$$

Adiabatic expansion: $PV^\gamma = \text{constant}$

Mean free path: $\ell = \frac{v_{rms} t}{(v_{rel})_{rms} t \pi d^2 n_V} = \frac{1}{\sqrt{2} \pi d^2 n_V}$