Chapter 31

Alternating Current Circuits
Alternating Current Circuits

- Alternating Current - Generator
- Wave Nomenclature & RMS
- AC Circuits: Resistor; Inductor; Capacitor
- Transformers - not the movie
- LC and RLC Circuits - No generator
- Driven RLC Circuits - Series
  - Impedance and Power
- RC and RL Circuits - Low & High Frequency
- RLC Circuit - Solution via Complex Numbers
- RLC Circuit - Example
- Resonance
Generators

By turning the coils in the magnetic field an emf is generated in the coils thus turning mechanical energy into alternating (AC) power.
Generators

Rotating the Coil in a Magnetic Field Generates an Emf

- Examples: Gasoline generator
- Manually turning the crank
- Hydroelectric power
Generators

\[ \varphi_m = NBA \cos \theta \quad \theta = \omega t \]
\[ \varphi_m = NBA \cos \omega t \]
\[ \mathcal{E} = - \frac{d}{dt} \varphi_m = NBA \omega \sin \omega t \]
\[ \mathcal{E} \left( \text{peak} \right) = \mathcal{E} \sin \omega t; \quad \mathcal{E} \left( \text{peak} \right) = NBA \omega \]
Wave Nomenclature and RMS Values
Wave Nomenclature

\[ A_{\text{peak-peak}} = A_{\text{p-p}} = 2A_{\text{peak}} = 2A_p; \quad A_p = \frac{A_{\text{p-p}}}{2} \]
Shifting Trig Functions

\[ x = A \left\{ \frac{\sin}{\cos} \right\} \left[ \omega t - \varphi \right] \]

The minus sign means that the phase is shifted to the right.

\[ x = A \left\{ \frac{\sin}{\cos} \right\} \left[ 2\pi \frac{t}{T} - \varphi \right] \]

A plus sign indicated the phase is shifted to the left.

\[ x = A \sin \left[ \omega t - \frac{\pi}{2} \right] \]

\[ x = A \left( \sin \omega t \cos \frac{\pi}{2} - \sin \frac{\pi}{2} \cos \omega t \right) \]

\[ x = A \left( \sin \omega t (0) - (1) \cos \omega t \right) \]

\[ x = -A \cos \omega t \]
Shifting Trig Functions

\[
sin \left[ \omega t - \frac{\pi}{2} \right] = 0
\]

\[
\omega t - \frac{\pi}{2} = 0
\]

\[
\omega t = \frac{\pi}{2}
\]

\[
t = \frac{\pi}{2\omega} ; \quad \frac{1}{\omega} = \frac{T}{2\pi}
\]

\[
t = \frac{\pi}{2} \frac{T}{2\pi} = \frac{T}{4}
\]
Root Mean Squared

The root mean squared (rms) method of averaging is used when a variable will average to zero but its effect will not average to zero.

Procedure

• Square it (make the negative values positive)
• Take the average (mean)
• Take the square root (undo the squaring operation)
Root Mean Squared Average

Sine Functions

Angle (Radians)

RMS Value

SIN(Theta)

SIN2(Theta)

MFMcGraw-PHY 2426
Chap31-AC Circuits-Revised: 6/24/2012
Average of a Periodic Function

\[ \langle V \rangle = V_{\text{avg}} = \frac{1}{T} \int_{0}^{T} V(t) dt; \quad V(t) = V_p \sin \omega t \]

\[ V_{\text{avg}} = \frac{1}{T} \int_{0}^{T} V_p \sin \omega t dt = \frac{V_p}{\omega T} \int_{0}^{\omega T} \sin x dx = - \frac{V_p}{\omega T} \int_{\cos(0)}^{\cos(\omega T)} d(\cos x) \]

\[ V_{\text{avg}} = - \frac{V_p}{\omega T} (1 - 1) = 0 \]
Root Mean Squared

\[ \langle V^2 \rangle = \left( V^2 \right)_{avg} = \frac{1}{T} \int_0^T V^2(t) dt; \quad V(t) = V_p \sin \omega t \]

\[ \left( V^2 \right)_{avg} = \frac{V_p^2}{T} \int_0^T \sin^2 \omega t dt = \frac{V_p^2}{\omega T} \quad \pi = \frac{V_p^2}{2} \]

\[ \left( V^2 \right)_{avg} = \frac{V_p^2}{2} \]

\[ V_{RMS} \equiv \sqrt{\left( V^2 \right)_{avg}} = \frac{1}{\sqrt{2}} V_p = 0.707V_p \]
The RMS voltage ($V_{RMS}$) is the DC voltage that has the same effect as the actual AC voltage.
RMS Power

\[ P_{\text{avg}} = \frac{1}{2} V_p I_p \]

since \( V_{\text{RMS}} = \frac{V_p}{\sqrt{2}} \) and \( I_{\text{RMS}} = \frac{I_p}{\sqrt{2}} \)

\[ P_{\text{avg}} = \frac{1}{2} \left( \sqrt{2} V_{\text{RMS}} \right) \left( \sqrt{2} I_{\text{RMS}} \right) \]

\[ P_{\text{avg}} = V_{\text{RMS}} I_{\text{RMS}} \]

The average AC power is the product of the DC equivalent voltage and current.
Resistor in an AC Circuit
Resistor in an AC Circuit

For the case of a resistor in an AC circuit the $V_R$ across the resistor is in phase with the current $I$ through the resistor.

In phase means that both waveforms peak at the same time.
Resistor in an AC Circuit

\[ P(t) = I^2(t)R = \left( I_p \cos \omega t \right)^2 R \]

\[ P(t) = I_p^2 R \cos^2 \omega t \]

The instantaneous power is a function of time. However, the average power per cycle is of more interest.

\[ P_{avg} = \frac{1}{T} \int_0^T P(t) \, dt \]

\[ P_{avg} = \frac{I_p^2 R}{T} \int_0^T \cos^2 \omega t \, dt \]

\[ P_{avg} = \frac{I_p^2 R}{T} \int_0^T \cos^2 \omega t \, dt = \frac{I_p^2 R}{2 \omega T} \pi = \frac{1}{2} I_p^2 R = \left( \frac{I_p}{\sqrt{2}} \right)^2 R = I_{RMS}^2 R \]
Inductors in an AC Circuit
# Coils & Caps in an AC Circuit

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Inductors in an AC Circuit

\[ \varepsilon_{peak} \cos \omega t = V_{L,peak} \cos \omega t = L \frac{dI}{dt} \]

\[ I = \frac{V_{L,peak}}{L} \int \cos \omega t \, dt = \frac{V_{L,peak}}{\omega L} \sin \omega t \]

\[ I = I_p \sin \omega t = I_p \cos \left( \omega t - \frac{\pi}{2} \right) \]

For the case of an inductor in an AC circuit the \( V_L \) across the inductor is 90° ahead of the current \( I \) through the inductor.
Inductors in an AC Circuit

\[ I = I_p \sin \omega t = \frac{V_{L \text{ peak}}}{\omega L} \cos (\omega t - \frac{\pi}{2}) \]

\[ I_p = \frac{V_{L \text{ peak}}}{\omega L} = \frac{V_{L \text{ peak}}}{X_L} \]

\[ X_L = \omega L \]

\[ X_L \] is the inductive reactance
Average Power - Inductors

\[ P(t) = V_L I = \left( V_{L\text{ peak}} \cos \omega t \right) \left( I_p \sin \omega t \right) \]

\[ P(t) = V_{L\text{ peak}} I_p \cos \omega t \sin \omega t \]

\[ P_{\text{avg}} = \frac{1}{T} \int_0^T V_{L\text{ peak}} I_p \cos \omega t \sin \omega t \, dt \]

\[ P_{\text{avg}} = \frac{V_{L\text{ peak}} I_p}{T} \int_0^T \cos \omega t \sin \omega t \, dt \]

\[ P_{\text{avg}} = \frac{V_{L\text{ peak}} I_p}{2T} \int_0^T \sin 2\omega t \, dt = 0 \]

Inductors don’t dissipate energy, they store energy.
Average Power - Inductors

Inductors don’t dissipate energy, they store energy.

The voltage and the current are out of phase by 90°.

As we saw with Work, energy changed only when a portion of the force was in the direction of the displacement.

In electrical circuits energy is dissipated only if a portion of the voltage is in phase with the current.
Capacitors in an AC Circuit
Capacitors in an AC Circuit

\[ V_C = \mathcal{E}_p \cos \omega t = V_{c_p} \cos \omega t \]

\[ Q = V_C C = V_{c_p} C \cos \omega t = Q_p \cos \omega t \]

\[ I = \frac{dQ}{dt} = -\omega Q_p \sin \omega t = -I_p \sin \omega t \]

\[ I = -\omega Q_p \sin \omega t = I_p \cos \left( \omega t + \frac{\pi}{2} \right) \]

For the case of a capacitor in an AC circuit the \( V_C \) across the capacitor is 90° behind the current \( I \) on the capacitor.
Capacitors in an AC Circuit

\[ I_p = \omega Q_p = \omega CV_{cp} = \frac{V_{cp}}{\frac{1}{\omega C}} = \frac{V_{cp}}{X_C} \]

\[ X_C = \frac{1}{\omega C} \]

\( X_C \) is the capacitive reactance.
Electrical Transformers
Electrical Transformers
Electrical Transformers
Electrical Transformers
Electrical Transformers

Both coils see the same magnetic flux and the cross sectional areas are the same

\[ B = \mu_0 n I \]

\[ \mu_0 n_1 I_1 = \mu_0 n_2 I_2 \]

\[ n_1 I_1 = n_2 I_2 \]

\[ I_2 = \frac{n_1}{n_2} I_1 \]

\[ \frac{I_1}{I_2} = \frac{n_2}{n_1} = \frac{N_2}{N_1} \frac{1}{L} = \frac{N_2}{N_1} \frac{1}{L} \]
Electrical Transformers

Conservation of Energy

Primary Power = Secondary Power

\[ V_{in}I_1 = V_{out}I_2 \]

\[ \frac{V_{out}}{V_{in}} = \frac{I_1}{I_2} = \frac{N_2}{N_1} \]

Induced voltage/loop

More loops => more voltage

Voltage steps up but the current steps down.
LC and RLC Circuits Without a Generator
LC Circuit - No Generator

To start this circuit some energy must be placed in it since there is no battery to drive the circuit. We will do that by placing a charge on the capacitor.

Since there is no resistor in the circuit and the resistance of the coil is assumed to be zero there will not be any losses.
LC Circuit - No Generator

Apply Kirchhoff’s rule

\[ L \frac{dI}{dt} + \frac{Q}{C} = 0 \]

Since \( I = \frac{dQ}{dt} \)

\[ L \frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \]

\[ \frac{d^2Q}{dt^2} = -\frac{1}{LC} Q \]

\[ \omega_R = \frac{1}{\sqrt{LC}} \]

This is the harmonic oscillator equation
LC Circuit - No Generator

\[ Q(t) = Q_p \cos \omega t \]

\[ I(t) = \frac{dQ}{dt} = -\omega Q_p \sin \omega t \]

\[ I(t) = -\omega Q_p \cos \left( \omega t + \frac{\pi}{2} \right) \]

The circuit will oscillate at the frequency \( \omega_R \). Energy will flow back and forth from the capacitor (electric energy) to the inductor (magnetic energy).
RLC Circuit - No Generator

Like the LC circuit some energy must initially be placed in this circuit since there is no battery to drive the circuit. Again we will do this by placing a charge on the capacitor.

Since there is a resistor in the circuit now there will be losses as the energy passes through the resistor.
RLC Circuit - No Generator

Apply Kirchhoff’s rule

\[ L \frac{dI}{dt} + IR + \frac{Q}{C} = 0; \quad I = \frac{dQ}{dt} \]

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = 0 \]

Damping term - friction

Restoring force “kx”

“ma” term

The damping term causes a damping of the natural oscillations of the circuit.
RLC Circuit - No Generator
RLC Circuit - No Generator

\[ L \frac{dI}{dt} + RI + \frac{Q}{C} = 0 \quad \text{Multiply by I} \]

\[ LI \frac{dI}{dt} + I^2R + \frac{QI}{C} = 0 \]

\[ \frac{d}{dt} \left[ \frac{1}{2} LI^2 \right] + I^2R + \frac{d}{dt} \left[ \frac{1}{2} \frac{Q^2}{C} \right] = 0 \]

\[ \frac{d}{dt} \left[ \frac{1}{2} LI^2 + \frac{1}{2} \frac{Q^2}{C} \right] = -I^2R \]

The rate of change of the stored energy = Power dissipated in the resistor
Series RLC Circuit with Generator

We have already examined the components in this circuit to understand the phase relations of the voltage and current of each component.

Now we will examine the power relationships.
Series RLC Circuit with Generator

\[ E(t) = E_{\text{peak}} \sin \omega t = E_p \sin \omega t \]

Apply Kirchhoff’s Loop rule to the circuit

\[ R I(t) + L \frac{dI(t)}{dt} + \frac{Q(t)}{C} = E(t) \]

\[ \frac{dQ}{dt} = I \quad \Rightarrow \quad Q(t) = Q_o + \int_0^t I(t')dt' \]

\[ R I(t) + L \frac{dI(t)}{dt} + \frac{1}{C} \int_0^t I(t')dt' = E(t); \text{ with } Q_o = 0 \]
Series RLC Circuit with Generator

\[ RI(t) + L \frac{dI(t)}{dt} + \frac{1}{C} \int_0^t I(t') dt' = \mathcal{E}(t) \]

Steady state \( \Rightarrow I(t) = I_p \sin \omega t \)

\[ \frac{dI(t)}{dt} = \omega I_p \cos \omega t; \quad \int_0^t I(t') dt' = \int_0^t I_p \sin \omega t' dt' = -\frac{I_p}{\omega} \cos \omega t \]

\[ RI_p \sin \omega t + \omega LI_p \cos \omega t - \frac{1}{\omega C} I_p \cos \omega t = \mathcal{E}_p \sin \omega t \]
Series RLC Circuit with Generator

\[ R I_p \sin \omega t + \omega L I_p \cos \omega t - \frac{1}{\omega C} I_p \cos \omega t = \mathcal{E}_p \sin \omega t \]

Change all “cos” to “sin” by shifting the angle

\[ R I_p \sin \omega t + \omega L I_p \sin \left( \omega t + \frac{\pi}{2} \right) + \frac{1}{\omega C} I_p \sin \left( \omega t - \frac{\pi}{2} \right) = \mathcal{E}_p \sin \omega t \]

The inductive voltage is 90° ahead of the current
The capacitive voltage is 90° behind of the current
Impedance in a Series RLC Circuit

\[ R I_p \sin \omega t + \omega L I_p \sin (\omega t + \frac{\pi}{2}) + \frac{1}{\omega C} I_p \sin (\omega t - \frac{\pi}{2}) = E_p \sin \omega t \]

The coefficients are voltages

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>L</th>
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<tbody>
<tr>
<td>RI_p</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>\omega LI_p</td>
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<td></td>
<td></td>
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<td>I_p/\omega C</td>
<td></td>
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<tr>
<td>X_L I_p</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>X_C I_p</td>
<td></td>
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<tr>
<td>X_L = \omega L</td>
<td>X_C = 1/\omega C</td>
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The R and X_L and X_C values are called impedances. That is a generalized term for resistance since they all have units of ohms.

X_L is the inductive reactance.

X_C is the capacitive reactance.
Power in a Series RLC Circuit

Now we go back to the original equation and multiply by 

\[ I(t) = I_p \sin \omega t \]

and integrate over one cycle: \( 0 \Rightarrow T \)

\[
RI_p \sin \omega t + \omega LI_p \cos \omega t - \frac{1}{\omega C} I_p \cos \omega t = E_p \sin \omega t
\]

\[
RI_p^2 \int_0^T \sin^2 \omega t dt + \omega LI_p^2 \int_0^T \sin \omega t \cos \omega t dt - \frac{I_p^2}{\omega C} \int_0^T \sin \omega t \cos \omega t dt = E_p I_p \int_0^T \sin^2 \omega t dt
\]

\[
\int_0^T \sin^2 \omega t dt = \pi \quad \int_0^T \sin \omega t \cos \omega t dt = 0
\]
Power in a Series RLC Circuit

\[ R I_{p}^2(\pi) + \omega LI_{p}^2(0) - \frac{I_{p}^2}{\omega C}(0) = E_{p} I_{p}(\pi) \]

- Power is only dissipated in the resistor.
- The inductor stores energy in its magnetic field.
- The capacitor stores its energy in its electric field.
Series RLC Circuit with Generator

We have used this equation to demonstrate the behavior of the three types of components: R, L and C, but-

We still haven’t solved the equation

\[ RI(t) + L \frac{dI(t)}{dt} + \frac{1}{C} \int_0^t I(t')dt' = \mathcal{E}(t); \quad \text{with } Q_0 = 0 \]

Before we actually solve it we need to introduce complex variables that will be used in the solution.
The RC Circuit
The RC Circuit - Low Freq

\[ RI_p \sin \omega t + \omega LI_p \cos \omega t - \frac{1}{\omega C} I_p \cos \omega t = \mathcal{E}_p \sin \omega t \]

Let \( L \Rightarrow 0 \)

\[ RI_p \sin \omega t - \frac{1}{\omega C} I_p \cos \omega t = \mathcal{E}_p \sin \omega t \]

For \( \omega t \approx 0; \ \cos \omega t \approx 1; \ \sin \omega t \approx 0 \)

\[ -\frac{1}{\omega C} I_p = \mathcal{E}_p(0) \]

\( I_p = 0 \Rightarrow \text{Open circuit} \)
The RC Circuit - High Freq

\[ R I_p \sin\omega t - \frac{1}{\omega C} I_p \cos\omega t = \mathcal{E}_p \sin\omega t \]

For \( \omega \gg \frac{1}{CR} \)

\[ I_p \sin\omega t - \frac{1}{\omega CR} I_p \cos\omega t = \frac{\mathcal{E}_p}{R} \sin\omega t \]

\[ I_p \sin\omega t = \frac{\mathcal{E}_p}{R} \sin\omega t \]

\[ I_p = \frac{\mathcal{E}_p}{R} \]

At high frequency the cap acts as a short circuit.

At high frequency the cap acts as a short circuit.
The RL Circuit
The RL Circuit - Low Freq

\[ RI_p \sin \omega t + \omega LI_p \cos \omega t - \frac{1}{\omega C} I_p \cos \omega t = \mathcal{E}_p \sin \omega t \]

Let \( C \rightarrow 0 \)

\[ RI_p \sin \omega t + \omega LI_p \cos \omega t = \mathcal{E}_p \sin \omega t \]

For \( \omega L \ll 1 \)

\[ RI_p \sin \omega t = \mathcal{E}_p \sin \omega t \]

\[ RI_p = \mathcal{E}_p \]

At low frequency L acts as a short circuit.
The RL Circuit - High Freq

\[ R I_p \sin \omega t + \omega LI_p \cos \omega t = E_p \sin \omega t \]

For \( \omega L \gg 1 \)

\[ \omega LI_p \cos \omega t = E_p \sin \omega t \]

multiply by \( \cos \omega t \) and average

\[ \omega LI_p \frac{1}{T} \int_{0}^{T} \cos^2 \omega t dt = E_p \frac{1}{T} \int_{0}^{T} \sin \omega t \cos \omega t dt \]

\[ \omega LI_p \pi = 0 \]

\[ I_p = 0 \Rightarrow Open \ circuit \]

At high frequency \( L \) acts as an open circuit.
Coils & Caps in an AC Circuit

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Complex Numbers for AC Circuits
Complex Numbers for AC Circuits

The basic complex (imaginary) number is “i.”

To avoid confusion we replace “i” with “j”

\[ j = \sqrt{-1} \]

\[ j^2 = jj = \sqrt{-1}\sqrt{-1} = -1 \]

\[ j^3 = jj^2 = j(-1) = -j \]

\[ j^4 = j^2 j^2 = (-1)(-1) = +1 \]

\[ j^5 = jj^4 = j(+1) = j \]
Complex Numbers for AC Circuits

Let a and b be real numbers

Then z is a complex number and $z^*$ is the complex conjugate

$$z = a + bj$$
$$z^* = a - bj$$
Complex Numbers for AC Circuits

The magnitude of \( z \)

\[
\text{Magn}(z) = |z| = \sqrt{(z^*)z} = \sqrt{(a - bj)(a + bj)}
\]

\[
|z| = \sqrt{a^2 + abj - abj - j^2b^2}
\]

\[
|z| = \sqrt{a^2 + b^2}
\]

\[
\tan \theta = \frac{b}{a}; \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)
\]

\[
z = |z|(\cos \theta + jsin \theta) = |z|e^{j\theta}
\]

The exponential representation of a complex number will prove useful in solving the RLC differential eqn.
Complex Numbers for AC Circuits

\[ z = |z|(\cos\theta + j\sin\theta) = |z|e^{j\theta} \]

\( e^{j\theta} \) can be viewed as a rotation operator in a complex space

\[ e^{j\pi/2} = j \]
\[ e^{j\pi} = -1 \]
\[ e^{j3\pi/2} = -j \]
\[ e^{j2\pi} = e^0 = +1 \]
Why Complex Numbers?

\[ z = |z|(\cos \theta + j \sin \theta) = |z|e^{j\theta} \]

Complex numbers simplify the solution of the integral-differential equations encountered in series RLC AC circuits.

The use of complex numbers simplifies the lead-lag nature of the voltage and current in AC circuits.
**Phasor Notation**

This diagram depicts a series RLC circuit driven at a frequency that causes the inductive voltage to be greater than the capacitive voltage. This gives the circuit an overall inductive nature - the current (in phase with $V_R$) is lagging the applied voltage $V_{app}$.

All of these voltage vectors (phasors) have a common time component ($e^{j\omega t}$) and so they all rotate at this common frequency. By suppressing this common rotation the concepts are easier to understand.
RLC Circuit Solution

\[ RI(t) + L \frac{dI(t)}{dt} + \frac{1}{C} \int_0^t I(t')dt' = E(t) \]

\[ I(t) = I_p e^{j\omega t} \]

The solution of a differential equation begins with the selection of a trial solution

\[ \frac{dI}{dt} = j\omega I_p e^{j\omega t} = j\omega I \]

\[ \int I(t)dt = \frac{I}{j\omega} \]

\[ RI + j\omega LI + \frac{I}{j\omega C} = E \]

\[ \left[ R + j\omega L - \frac{j}{\omega C} \right] I = E \]

\[ R + j \left[ \omega L - \frac{1}{\omega C} \right] \frac{E}{I} \]
RLC Circuit Solution

\[
R + j \left[ \omega L - \frac{1}{\omega C} \right] = \frac{E}{I}
\]

\[Z = R + j \left[ \omega L - \frac{1}{\omega C} \right] = \frac{E}{I}\]

These are complex variables

The quantity \(Z\) is called the impedance and it is a complex variable

\[E = I Z\] is a complex version of Ohm’s Law
Complex Impedance

\[ Z = R + j \left[ \omega L - \frac{1}{\omega C} \right] \]

\[ |z| = \sqrt{(z^*)z} = \sqrt{R^2 + \left[ \omega L - \frac{1}{\omega C} \right]^2} \]

\[ \tan \theta = \frac{X_L - X_C}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} \]

\[ X_L = \omega L \]

\[ X_C = \frac{1}{\omega C} \]
Relative Voltage Phases - Inductive

Impedance Space

\[ X_L = \omega L \]

\[ X_C = \frac{1}{\omega C} \]

Voltage Space
Phases in an Inductive AC Circuit

In an inductive circuit the voltage peaks first and the current peaks later.

All vectors rotating at a common frequency $\omega$

In an inductive circuit the voltage peaks first and the current peaks later.
Phases in a Capacitive AC Circuit

In capacitive circuit the current peaks first and the voltage peaks later.
RLC Series AC Circuit Example
R = 250Ω, L = 1.20mH, 
C = 1.80µF, V_p = 120v, f = 60Hz

Determine the following:
(a.) X_L - Inductive reactance
(b.) X_C - Capacitive reactance
(c.) Z - Impedance
(d.) θ - Phase angle
(e.) I_p - Peak current
(f.) I_{RMS} - RMS current
(g.) ω_R - Resonance frequency
(h.) P_{avg} - Average Power
R = 250Ω, L = 1.20mH,
C = 1.80µF, V_p = 120v, f = 60Hz

Determine the following:
(a.) \( X_L \) - Inductive reactance
(b.) \( X_C \) - Capacitive reactance
(c.) \( Z \) - Impedance

First calc: \( \omega = 2\pi f = 2(3.14)60 = 377 \text{ rad/s} \)

\[ X_L = \omega L = 377(1.20\times10^{-3}) = 0.452\Omega \]

\[ X_C = \frac{1}{\omega C} = \frac{1}{((377)(1.80\times10^{-6}))} = 1474\Omega \]

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{250^2 + (0.452 - 1474)^2} \]

\[ Z = 1495 \Omega \]
\[ X_L = \omega L \]
\[ X_C = \frac{1}{\omega C} \]
\[ Z = \sqrt{R^2 - (X_L - X_C)^2} \]

\[ R = 250\Omega, \quad L = 1.20\text{mH}, \]
\[ C = 1.80\mu\text{F}, \quad V_p = 120\text{v}, \quad f = 60\text{Hz} \]

Determine the following:
(d.) \( \theta \) - Phase angle

\[ \theta = \tan^{-1} \left[ \frac{X_L - X_C}{R} \right] = \tan^{-1} \left[ \frac{0.452 - 1474}{250} \right] = \tan^{-1} \left[ \frac{-1474}{250} \right] = -80.4^0 \]
R = 250Ω, L = 1.20mH,
C = 1.80µF, \( V_p = 120v \), \( f = 60Hz \)

Determine the following:

(e.) \( I_p \) - Peak current

(f.) \( I_{RMS} \) - RMS current

\[
I = \frac{V}{Z} = \frac{V_p e^{j\omega t}}{Z_p e^{j\theta}} = \frac{V_p}{Z_p} e^{j(\omega t - \theta)} = I_p e^{j(\omega t - \theta)}
\]

\[
I_p = \frac{V_p}{|Z|} = \frac{120}{1495} = 80.3mA
\]

\[
I = I_p e^{j(\omega t - \theta)} = 80.3mA e^{j(\omega t + 80.4)}
\]

\[
I_{RMS} = \frac{I_p}{\sqrt{2}} = 0.707I_p = 56.7mA
\]
R = 250Ω, L = 1.20mH, 
C = 1.80µF, \( V_p = 120v \), \( f = 60Hz \)

Determine the following:

(g.) \( \omega_R \) - Resonance frequency

\[
\omega_R = \frac{l}{\sqrt{LC}} = \frac{l}{\sqrt{(1.20 \times 10^{-3}) (1.80 \times 10^{-6})}} = 21.5 krad/s
\]

\[
f = \frac{\omega}{2\pi} = \frac{21.5 \times 10^3}{6.28} = 3.42 kHz
\]
R = 250Ω, L = 1.20mH,
C = 1.80µF, \( V_p = 120v \), f = 60Hz

Determine the following:

(h.) \( P_{avg} \) - Average Power

\[
S = \frac{1}{2} VI \quad V = V_p e^{j\omega t} \quad I = I_p e^{j(\omega t+80.4)}
\]

\[
S = \frac{1}{2} \left( V_p e^{j\omega t} \right) \left( I_p e^{-j(\omega t+80.4)} \right) = \frac{1}{2} V_p I_p e^{j(\omega t-\omega t-80.4)}
\]

\[
S = \frac{1}{2} V_p I_p e^{-j(80.4)} = \frac{V_p I_p}{\sqrt{2}} e^{-j(80.4)} = V_{RMS} I_{RMS} e^{-j(80.4)}
\]
Determine the following: (h.) $P_{avg}$ - Average Power

\[
S = \frac{1}{2} V_p I_p e^{-j(80.4)} = \frac{V_p}{\sqrt{2}} \frac{I_p}{\sqrt{2}} e^{-j(80.4)} = V_{RMS} I_{RMS} e^{-j(80.4)}
\]

\[
S = V_{RMS} I_{RMS} (\cos(80.4) - j\sin(80.4))
\]

\[
S = V_{RMS} I_{RMS} \cos(80.4) - jV_{RMS} I_{RMS} \sin(80.4)
\]

\[
P_{avg} = V_{RMS} I_{RMS} \cos(80.4)
\]

Power Factor

\[
P_{avg} = 84.3 \left(56.7 \times 10^{-3}\right) (0.167) = 0.803W
\]
Voltages

\[ V_R = I_{\text{RMS}} R = (56.7 \times 10^{-3})(250) = 14.2 \text{V} \]
\[ V_L = I_{\text{RMS}} X_L = (56.7 \times 10^{-3})(377)(1.20 \times 10^{-3}) = 0.0265 \text{V} \]
\[ V_C = I_{\text{RMS}} X_C = (56.7 \times 10^{-3})/\left(377(1.80 \times 10^{-6})\right) = 83.6 \text{V} \]

\[ V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(14.2)^2 + (0.0256 - 83.6)^2} \]
\[ V = 84.8 = V_{\text{RMS}} \]
Resonance in a Series RLC Circuit

Fig. 8-2. Series Circuit $Z$, $\theta$ and $Y$ as Functions of $\omega$. 
Resonance in a Series RLC Circuit
Power Transfer and Resonance

\[ L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V_p \cos \omega t \]

\[ P_{avg} = I_{RMS}^2 R = \frac{V_p^2}{Z^2} R \]

\[ Z^2 = R^2 + \left( \omega L - \frac{I}{\omega C} \right)^2 \]

\[ P_{avg} = \frac{V_p^2}{R} \left( I + \left( \frac{\omega L}{R} \right)^2 \left[ \frac{\omega^2 - \omega_0^2}{\omega^2} \right]^2 \right) \]

\[ Z = \sqrt{R^2 + \left( X_L - X_C \right)^2} \]

\[ I = I_p \cos \left( \omega t - \delta \right) = \frac{V_p}{|Z|} \cos \left( \omega t - \delta \right) \]
**Q Factor = Measure of Stored Energy**

\[
P_{\text{avg}} = \frac{V_p^2}{R} \left(1 + \left[\frac{\omega L}{R}\right]^2 \left[\frac{\omega^2 - \omega_0^2}{\omega^2}\right]^2\right)
\]

Q-Factor

\[
Q = 2\pi \frac{E}{\Delta E} = \frac{\omega_0}{\Delta \omega} = \frac{f_0}{\Delta f}
\]

\(\Delta \omega = \text{FWHM}\)

FWHM = Full Width at Half Maximum

As an approximation

\[
Q \approx \frac{\omega_0 L}{R}
\]

E = Total Energy and
\(\Delta E\) is the dissipated energy
Resonance in a Series RLC Circuit

\[ P_{av} \]

\[ P_{av\,\text{max}} \]

\[ \frac{1}{2} P_{av\,\text{max}} \]

\[ P'_{av\,\text{max}} \]

\[ \frac{1}{2} P'_{av\,\text{max}} \]

\[ \Delta \omega \]

\[ \Delta \omega' \]

Small \( R \), large \( Q \)

Large \( R \), small \( Q \)

\[ \omega_0 \]

\[ \omega \]
RLC Parallel Circuit

We’re not covering this type of circuit
Extra Slides
When alternating voltage is applied to the primary winding, an alternating current will flow that will magnetize the magnetic core, first in one direction and then in the other direction. This alternating flux flowing around the entire length of the magnetic circuit induces a voltage in both the primary and secondary windings. Since both windings are linked by the same flux, the voltage induced per turn of the primary and secondary windings must be the same value and same direction. This voltage opposes the voltage applied to the primary winding and is called counter-electromotive force (CEMF).