

## **Measurements and Data Analysis**   Revised 5/20/2011

### **Significant Figures**

Computations

### **Types of Errors**

Random Errors & Systematic Errors

### **Deviation from the Mean**

Range - quick and very approximate

Absolute Deviation - poor man's Std Dev - before calculators

Standard Deviation - useful for random Gaussian errors that are typical of measurements.

### **Statement of Measurement Result**

$X_{\text{Meas}} = X_{\text{Avg}} \pm \Delta X$      where  $\Delta X$  signifies the deviation measure chosen.

### **Accuracy & Precision**

#### **Expressing measurement errors / uncertainty**

% Error	}	Accuracy related
% Difference		
% Uncertainty		Precision related

Instrument Limited Error - equipment limitation

### **Propagation of Error**

Errors in the result of a calculation are due to errors in the numbers used in performing the calculation.

Addition / Subtraction

Multiplication / Division

### **Data Analysis Example - Atwood's Machine**

Curve Fitting - Limited to Linear "Curves" or Trendlines, sometimes called Regression lines.

Least Squares curve fitting

Linear Trendline Equation: Slope and y-intercept

LINEST - Line statistics - Std Dev of slope and intercept

Data processing to produce a linear relationship:  $X \Rightarrow \log(X)$ ;  $T \Rightarrow T^2$

## Rules for Significant Digits:

### 1. All non-zero numbers count.

Ex. 135.72 has 5 significant digits

### 2. Zeros either count or don't count depending upon their position:

#### a. Leading zeros:

i. Before the decimal point – aren't necessary in terms of either value of the number or implied accuracy of the measurement so they don't count.

Ex. 000356. has 3 significant digits

ii. After the decimal point – are necessary in terms of the value of the number (i.e., they hold place), but otherwise they do not imply anything about the accuracy of the measurement so they don't count.

Ex. 0.00008539 has 4 significant digits

**b. In between zeros:** zeros in between other numbers always count no matter if they are in front of or after the decimal point, because they are part of the value of the number.

Ex. 20850.406 has 8 significant digits

#### c. Ending zeros:

i. Before the decimal point – are necessary because they both indicate the value of the number (i.e., they hold place) and they are also supposed to imply the accuracy of the measurement (i.e., they give significance), thus ending zeros may or may not play a dual role and so they are indeterminate!

Ex. 25000 has an indeterminate number of significant digits

Note: This number should be expressed in scientific notation in order to correctly convey to the reader the appropriate number of significant digits in this number, as follows: if the number has only 2 significant digits it should be written as  $2.5 \times 10^4$ , if the number has 3 significant digits it should be written as  $2.50 \times 10^4$ , if the number has 4 significant digits it should be written as  $2.500 \times 10^4$ , and finally if the number has 5 significant digits then it should be written as  $2.5000 \times 10^4$ !

ii. After the decimal point - aren't necessary in terms of the value of the number, but they do imply a difference in the accuracy in the measurement (i.e., they give significance) so they do count.

Ex. 67.4300 has 6 significant digits

Note: The number 67.4300 differs from the number 67.43 because of the implied accuracy in the measurement of these numbers. The implied accuracy is at most  $\pm 1$  in the right most significant digit. Thus, the implied uncertainty range for 67.4300 is 67.4299 – 67.4301 while the implied uncertainty range for 67.43 is 67.42 – 67.44. Notice that these vastly different ranges suggest that these two numbers are indeed different

## **Measurement Errors**

### **Random Errors -**

Random events are independent of each other, i.e. as in the flipping of a coin. The result of one flip does not influence the outcome of the next flip and is not itself influenced by the outcome of the previous flip. To better understand random events it is necessary for them to have a common probability distribution. The outcomes of all the coin flip trials belong to the same probability distribution. It is impossible to know the exact nature of the probability distribution without running an infinite number of trials but they can be estimated if a sufficiently large number of trials are conducted. The random errors that will be encountered in this class all belong to the family of Gaussian probability distributions - the famous bell-shaped curve. In events described by the Gaussian probability distribution extreme events are rare. The uncertainty in the experimental results can be quantified in the case of these random errors described by the Gaussian probability distribution.

### **Systematic Errors -**

These types of errors can show up as any of the following

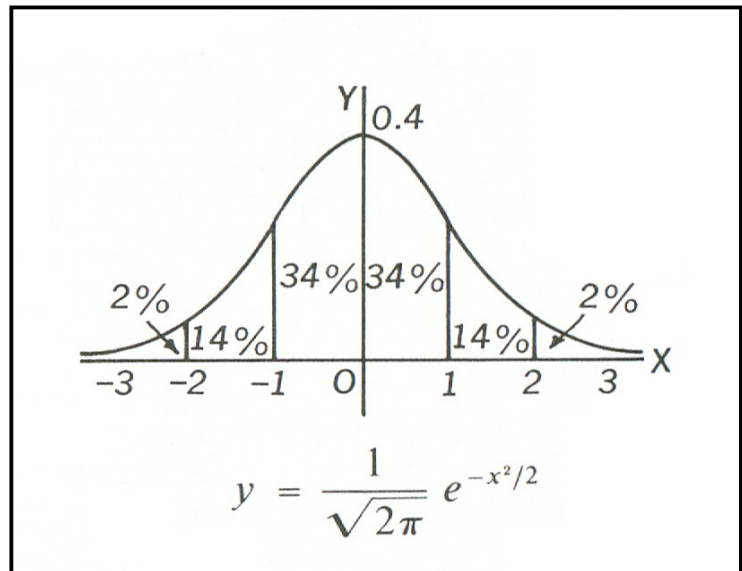
- Variations in the physical property being measured.
- Variations in the environmental variables: temperature, humidity, etc.
- Variation in the reading of the measurement instrument.
- Imprecision in the experimental equipment.

These can be the result of equipment malfunctions or design errors. They can also be caused by incorrect experimental procedures. Systematic errors can also arise from the variation in certain variables that have been ignored in the theoretical description of the system being measured. These are usually ignored because they were believed to be unimportant. Sometimes systematic errors make their presence known and the size of the effect can be quantified. Systematic errors are eliminated or controlled in successive repetitions of the experiment.

Our goal this semester will be to analyze our measurements assuming random errors and in stating our results to close in on any possible systematic errors in the various lab experiments.

## Random Errors - Gaussian Distribution

The measurement errors typically encountered in the physics laboratory can be described by the Gaussian probability distribution (see diagram to the right). This diagram shows a normalized distribution with mean 0 and standard deviation of 1. The average of a series of measurements (of the same variable) is the best estimate of the mean describing the probability distribution. The standard deviation is a measure of the dispersion of the measurements about the mean. Together, the mean and the standard deviation are enough to totally describe the Gaussian distribution (this also sometimes



called a probability density). The vertical lines represent the positions, on either side of the mean, of distances that are  $\pm 1$ ,  $\pm 2$ ,  $\pm 3$  standard deviations from the mean. The area under the curve represents the probability. The total area represents a probability of 1 because  $X$  extends to infinity in both directions so every possible occurrence is represented. The area under the curve between the  $\pm 1$  standard deviations lines represents 68.3% of the total area. The area under the curve between the  $\pm 2$  standard deviations lines represents 95.4% of the total area. The area under the curve between the  $\pm 3$  standard deviations lines represents 99.7% of the total area.

This means that you can express the result of a series of measurements of the quantity  $X$ , using the average (mean) of those measurements and the standard deviation (StdDev), as shown below:

$$X_{\text{Meas}} = X_{\text{Avg}} \pm \text{StdDev}$$

If these series of measurements are truly from a population described by the Gaussian probability distribution then if you repeat the series of measurements there is a 68.3% probability (chance) that the average of this second series will fall within 1 StdDev of the first average. That is, the second average will be within interval defined by

$$X_{\text{Avg}} + \text{StdDev} \quad \text{and} \quad X_{\text{Avg}} - \text{StdDev}$$

68.3% of the time.

## Deviation from the Mean

Range - quick and very approximate

Absolute Deviation - poor man's Std Dev - before calculators

Standard Deviation - useful for random Gaussian errors that are typical of measurements.

There is always some scatter in experimental measurements. Even when making multiple measurements, with extreme care, there will be some variation and it is of interest to quantify the degree of that scatter. This scattering is referred to as deviation from the mean and there are several ways to describe it. Some methods are quick and dirty and were favored in the era before handheld calculators due to their ease of calculation.

We want to be able to state our experimental results in the following form:

$$X_{\text{Meas}} = X_{\text{Avg}} \pm \Delta X \quad \text{where } \Delta X \text{ signifies the chosen deviation measure.}$$

### Range

The quickest estimate of the deviation from the mean is called the Range

$$\text{Range} = \text{ABS}(\text{Largest Data} - \text{Smallest Data})$$

It is just the absolute value of interval between the largest and the smallest data points. This is seldom used anymore.

### Absolute Deviation - poor man's Std Dev - before calculators

This is a kind of “mathed down” standard deviation. It avoids the squaring and the final square root that was a plus in the pre-calculator era.

$$\text{Absolute Deviation} = (1/N) * \sum \text{ABS} (X_i - X_{\text{Avg}})$$

The absolute values of the differences of each data point from the mean are summed. The result is then divided by N, the total number of measurements.

### The Standard Deviation

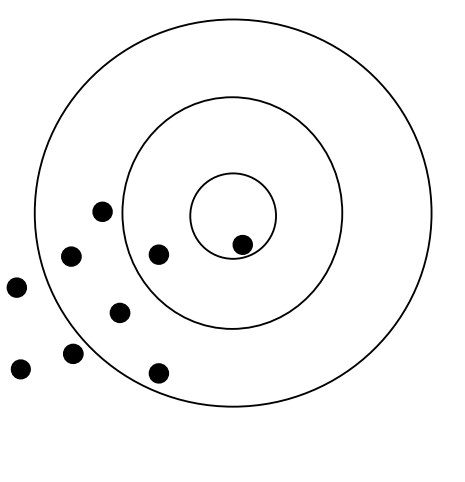
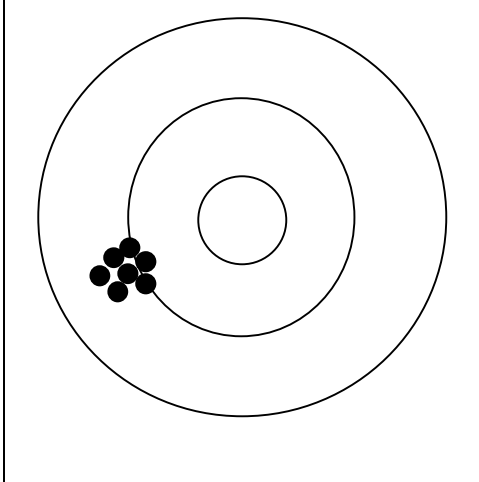
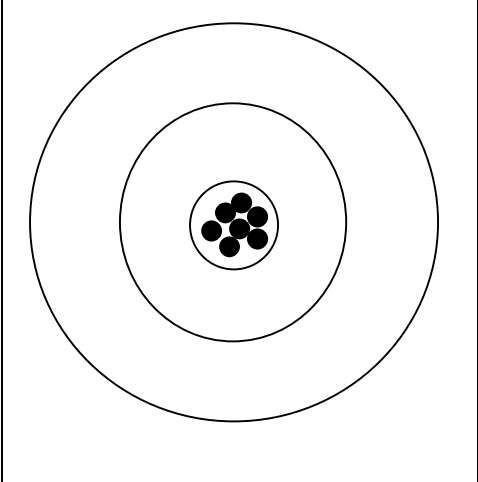
With the coming of calculators and spreadsheet the standard deviations is used almost exclusively in describing measurement errors.

$$\text{Standard Deviation} = \sigma = \text{SQRT} [(1/N-1) * \sum (X_i - X_{\text{Avg}})^2]$$

The presence of “N-1” in the denominator instead of “N” is for statistical reasons that won't be discussed here.

## Accuracy and Precision

The terms precision and accuracy carry very specific meanings in the world of physical measurements. They are also a source of much confusion for a student taking a first physics course or any science course requiring measurements. **Precision** is related to repeatability while **Accuracy** is related to a comparison to a standard or accepted value. This is best illustrated by the diagrams below, which represent bulls-eye targets, each consisting of three concentric rings. The dots are bullet holes in the targets and represent a series of experimental measurements. The quantity sought is the center of the inner circle. In the first target on the left the shots are scattered and only one has come close to the center of the inner circle. The scattering of the shots indicates that there is little precision (repeatability) in the measurements. The fact that only one reached the inner circle demonstrates the lack of accuracy. The center target diagram shows a tight grouping of the shots demonstrating a high degree of repeatability or precision. However, the center of the shot group is quite a distance from the center of the target and demonstrates a lack of accuracy. The shots in the target on the far right demonstrate both high precision and accuracy. The center of the tight group of shots is centered on the inner circle.

		
No Accuracy & No Precision	Precision but No Accuracy	Precision & Accuracy

## Expressing measurement errors / uncertainty

% Uncertainty	}	Precision related
% Error		Accuracy related
% Difference		
Instrument Limited Error - equipment limitation		

### Precision

The most basic way to reduce the effect of measurement errors is to repeat the measurement multiple times. How many times the measurement should be repeated does not have an all-purpose answer. The answer falls under the general principle of “Good enough for what it’s for.” The principle can be demonstrated by the following example. Suppose you are interested in knowing the width of a certain doorway. How you measure the doorway and how many times you repeat the measurement will depend on what the result of the measurement will be used for. If your interest is driven by pure curiosity then you might just look at the doorway and guess the width without performing any formal measurement. If you need to fit a piece of furniture through the doorway then you would probably use a tape measure to make a quick measurement. However, if you are planning on purchasing a door for that doorway then your interest in the exact width is substantially increased. You might even be concerned enough to make measurements in several places, in case the width is not constant over the entire height of the doorway.

Multiple measurements are the hallmark of an attempt to determine the precision of a series of measurements. Precision is the basic requirement for any measurement system - the ability to repeat the same measurement multiple times with the resulting measurements grouped tightly together. If you can’t get precision then worrying about accuracy is a waste of time.

To determine the precision it is necessary to calculate the average ( $X_{Avg}$ ) and the standard deviation (StdDev) of the measurements. The number of measurements taken is determined by many considerations. More measurements allow for a better determination of the average and standard deviation and might be possible if the measurements are not too expensive and don’t take too long. Sometimes it is as simple as having the lab handout specify how many measurements to make. The result of your calculations would yield the following statement of your results.

$$X_{Meas} = X_{Avg} \pm StdDev$$

The % Uncertainty is given by

$$\% \text{ Uncertainty} = StdDev * 100 / X_{Avg}$$

In the typical lab in this course the goal is usually to have your measurement precision at or below 1.0% if possible.

## Accuracy

Accuracy is related to the comparison of your results to a standard or to an accepted value.

$$\% \text{ Error} = (X_{\text{Avg}} - X_{\text{Std}}) * 100 / X_{\text{Std}}$$

where  $X_{\text{Std}}$  is the standard value for the quantity  $X$ . Some books use the absolute value to determine the value of the %Error but in doing so you lose the information as to whether your experimental value is above or below the standard or accepted value. Knowledge of this direction of variation is vital if the experiment is going to be improved for future use.

Sometimes a standard value isn't available and a comparison is made between two different measurements of the same quantity and it is desirable to quantify the results. In this case a slightly different quantity is calculated. This is called the % Difference. Here it cannot be determined if one measurement is better than the other so both must be treated equally. In the %Error calculation the  $X_{\text{Std}}$  value is given special treatment and appears in two places in the equation. Instead of using one or the other measurement in the denominator the average of the two measurements is used instead to avoid biasing the % Difference either up or down.

$$\begin{aligned} \% \text{ Difference} &= (X_{\text{Avg1}} - X_{\text{Avg2}}) * 100 / ((X_{\text{Avg1}} + X_{\text{Avg2}})/2) \\ &= (X_{\text{Avg1}} - X_{\text{Avg2}}) * 200 / (X_{\text{Avg1}} + X_{\text{Avg2}}) \end{aligned}$$

## Instrument Limited Error - equipment limitation

Every instrument used in performing a measurement has a smallest unit of measurement. On a meter stick it is the smallest distance between two adjacent scale lines - usually 0.10 cm or 1.0 mm. On a digital scale it might be 0.1 grams. In the case of the meter stick, if the item being measure has a well-defined edge it might be possible to estimate the fraction of the smallest meter stick interval, fractions of a millimeter. In the case of the digital scale all we know is that mass measurement is somewhere between the displayed number and that number plus or minus "0.1" grams. In either case, there is a residual uncertainty due to the scale limitation of the instrument itself. We describe this as the Instrument Limited Error. The most popular method of quantifying this error is to set it equal to  $\pm$  one-half the smallest division.

$$\text{Instrument Limited Error} = \text{ILE} = \pm \frac{1}{2} \text{ Smallest Division}$$



## **Propagation of Error**

It isn't possible to directly measure every quantity. Some quantities are derived by way of a calculation that is based on variables that were measured directly. The final resulting error in the calculation is due to errors in the numbers used in performing the calculation. The passing on of this error is referred to as the Propagation of Error. It is desirable to estimate the size of the error in the final result due to the errors in the contributing variables. The analysis can become quite complicated and relies heavily upon the use of Calculus. However, for our purposes, we only need to consider two categories that will cover most of the situations that will arise in a basic Physics course. These include the categories of formulas involving: Addition / Subtraction and Multiplication / Division. In the following we will assume that the variables to be used in the calculations are X, Y and Z. It is assumed that  $\Delta X$ ,  $\Delta Y$ , and  $\Delta Z$  signify the chosen deviation measures and  $X_{Avg}$ ,  $Y_{Avg}$ ,  $Z_{Avg}$  are the corresponding average values of X, Y and Z. SQRT is the square root function.

### **Addition / Subtraction**

Example:  $W = X + Y - Z$

$$\Delta W = \text{SQRT} [\text{ABS}(\Delta X) + \text{ABS}(\Delta Y) + \text{ABS}(\Delta Z) ]$$

### **Multiplication / Division**

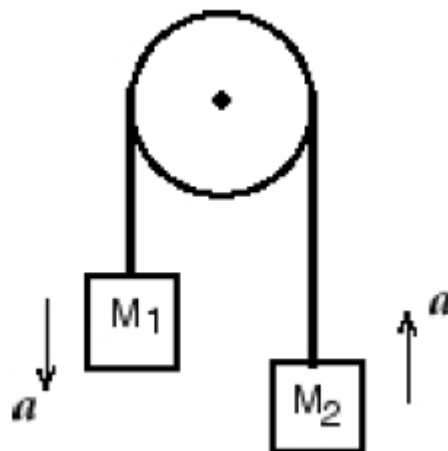
Example:  $W = X*Y/Z$

$$(\Delta W / W_{Avg}) = \text{SQRT} [(\Delta X / X_{Avg})^2 + (\Delta Y / Y_{Avg})^2 + (\Delta Z / Z_{Avg})^2 ]$$

## Atwood's Machine - Data Analysis Example

Atwood's Machine provides a means of obtaining a measurement of the acceleration due to gravity ("g"). The basic simplicity of the experimental set up can be appreciated from the diagram to the right. The equation below relates the measured acceleration "a" to "g", the acceleration due to gravity.

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$



The basic intent of the Atwood Machine is to slow down the motion so that the system acceleration is easier to measure. Then, by applying Newton's 2<sup>nd</sup> Law to the system, the above equation for the calculated acceleration can be derived, where friction has been explicitly ignored.

The table below shows some typical experimental data. The masses are in grams since they will only appear in the form of a ratio in the equation above. The accelerations are in units of m/s<sup>2</sup> and were measured with a smart pulley / photogate combination. In this version of Atwood's Machine the sum of the two masses, M<sub>1</sub> + M<sub>2</sub>, was NOT kept constant.

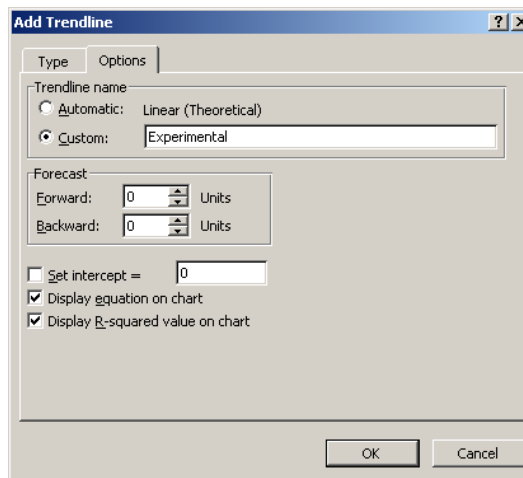
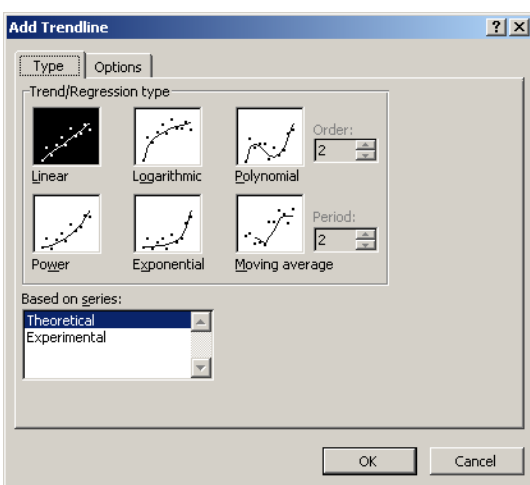
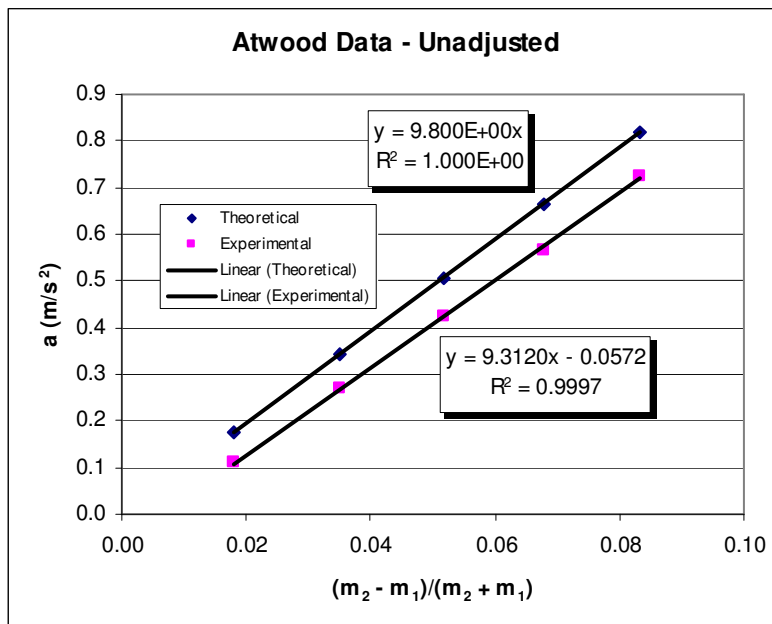
M <sub>1</sub>	M <sub>2</sub>	M <sub>1</sub> + M <sub>2</sub>	Del m	Delm/(M <sub>1</sub> +M <sub>2</sub> )	a <sub>Calc</sub>	a <sub>Meas</sub>
55.0	57.0	112.0	2.0	0.018	0.1750	0.1102
55.0	59.0	114.0	4.0	0.035	0.3439	0.2696
55.0	61.0	116.0	6.0	0.052	0.5069	0.4254
55.0	63.0	118.0	8.0	0.068	0.6644	0.5669
55.0	65.0	120.0	10.0	0.083	0.8167	0.7237

The value of "g" could be determined for each row in the above table but that would lead to inconsistent values each of which would contain significant errors. Our approach will be to use all the data and do a least squares linear fit to the data.

## The Linear Trend Line

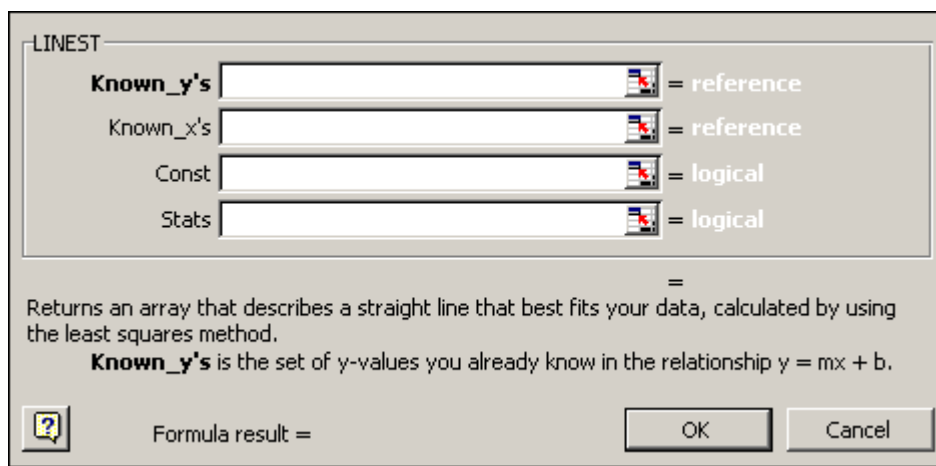
On the right are two sets of Atwood data plotted on a linear chart of acceleration versus the mass ratio  $(m_2 - m_1)/(m_1 + m_2)$ . The lower line is the data from the  $a_{\text{Meas}}$  column and the upper line is from the  $a_{\text{Calc}}$  column in the data table above. Linear trendlines were then drawn through the respective data points. Selecting “Add Trendline” will summon up the two-tabbed dialogue box shown below. Select the linear trendline on the “Type” page and on the “Options” page select the last two options: “Display equation on chart” and “Display R-squared value on the chart.” The fitting parameters are the slope and the y-intercept, represented by the x-coefficient and the constant term respectively in the trendline equation.

If the trendline function also displayed the uncertainty statistics of its fitting parameters we would be done here. But it doesn't so we need to use the LINEST worksheet function that is explained on the next page.



## LINEST Directions

In Excel you will need to use the LINEST Worksheet function to calculate the uncertainties in the slope and the y-intercept. Start by selecting a 2-column by 2 –row area of empty cells on the spreadsheet. Then, under the **Inset** menu item select the **Function** option – or – click on the “**f<sub>x</sub>**” button on one of the tool bars. The **Paste Function** dialogue box will appear. In the left window select the **Statistical** Function category. In the right window scroll down to find the **LINEST** function and select it. Then click OK. The **LINEST** dialogue box will appear with four input items: “Known y’s”, “Known x’s”, “Const”, “Stats”. Place the y and x cell ranges in the first two input boxes. If the “Const” item is TRUE or omitted the y-intercept will be calculated - which is what we want. DO NOT force the trendline to go through the origin. It will be set to zero if “Const” is set to FALSE. “Stats” is set to TRUE to get the array of statistics to be displayed in the 2x2 array. Now this is an **ARRAY** Function and the “Ctrl” and “Shift” keys need to be held down when pressing the “Enter” key or clicking the OK button.



Results format

slope	y-intercept
std error in slope	std error in y-intercept

Typical results

9.834	-0.08318
0.196611	0.009233

In the table below are the results of the LINEST worksheet function applied to the experimental data shown above. This allows you to analyze the precision or repeatability of your experiment.

	Slope	y-intercept
Value	9.31197	-0.05724
Uncertainty	0.09869	0.00554

$$\text{Slope} = 9.31 \pm 0.10 \text{ m/s}^2$$

$$\text{Percent Uncertainty} = \pm 1.1\%$$

$$\text{Y-intercept} = -0.0572 \pm 0.0055 \text{ m/s}^2$$

$$\text{Percent Uncertainty} = \pm 9.6\%$$

The  $\pm 1.1\%$  uncertainty in the value of the slope tells you that the experiment is very repeatable. If you repeat the experiment again there is a 68.3% chance that your slope value will fall in the interval 9.21 to 9.41  $\text{m/s}^2$  ( $9.31 \pm 0.10 \text{ m/s}^2$ ). This is a precision (repeatability) measure and depends on the experimental design, the equipment itself and your ability to use the equipment and to make measurements.

Next the accuracy of the results need to be examined. This requires the comparison of your results to a standard or to a known value. To compute the percent error you need to compare your experimental slope with known value of  $g = 9.80 \text{ m/s}^2$

$$\text{Percent Error} = (\text{Slope} - g) * 100 / g = -5.0\%$$

The result of the calculation shows that your experimental value is 5.0% below the accepted value of  $g$ . This is almost 5 times the value of the percentage uncertainty in your slope measurements. Therefore this is a statistically significant difference and cannot be explained away by bad experimental technique.

## Analysis

Where was the problem? Go back and list every quantity that you measured and estimate the uncertainty in those measurements. There were two basic measurements: masses and accelerations.

## Mass

A classic mistake is to forget to include the mass of the weight hanger but we will assume that wasn't the case here because your lab notes indicated that it was included. The masses are stable and don't change with time, their values are engraved into the masses themselves and an electronic scale is available that will determine the mass values to within  $\pm 0.05$  grams. For masses of  $\sim 50$  grams this represents an uncertainty of  $\pm 0.05 * 100 / 50 = \pm 0.10\%$ . So, the problem doesn't seem to be in the masses.

## Accelerations

The problem would appear to be in the determination of the *acceleration* of the Atwood system of masses. This variable was measured electronically using a Smart Pulley (with a photogate) and the Logger Pro data system. The distance is measured and the velocity and acceleration are calculated from the distance-time data. The rotation of the 10-spoked pulley breaks the infrared light beam in the photogate. Every time this basic event happens the data acquisition program adds one-tenth of the pulley circumference to the distance traveled by the masses, and records the time. The timing element is electronic and very precise which brings us to the "one-tenth of the circumference" number. There is a programmed-in value or the user can enter his/her own number. So the question becomes: How well is this number known? For the case of the string in the groove the default value is 0.015m or 15mm. Just using  $\pm$ one-half of the least significant digit as the measure of uncertainty gives a percent uncertainty of  $\pm 0.5 * 100 / 15 = \pm 3.3\%$ .

The accelerations are determined from the slope of the velocity versus time curves. The uncertainties in these acceleration slopes are determined by the Logger Pro software. These percent uncertainties are typically under 1%.

## Massless Pulley

Although it was assumed that the Smart Pulley was massless it actually has a mass of 5.5g. Using the value of the pulley's moment of inertia of  $1.8 \times 10^{-6} \text{ kg-m}^2$ , the mass 5.5g and its

radius of 2.50 cm, it is possible to determine that its  $\beta$  factor is 0.524 which is close to  $\frac{1}{2}$ . Hence it behaves as a solid disk.

If the mass of the pulley is included in the analysis the acceleration formula becomes.

$$a = \left( \frac{m_1 - m_2}{m_1 + m_2 + \left(\frac{M}{2}\right)} \right) g$$

Since the mass of the pulley is quite a bit smaller than the hanging masses we can approximate the acceleration by the expression below.

$$a = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \left( 1 - \frac{M/2}{m_1 + m_2} \right) g$$

This shows that the simple acceleration (massless pulley) is reduced by about 2.5%.

### Separate the Friction

The real culprit here with the experimental design. The fact that the total mass in the system was changing, i.e.  $(m_1 + m_2)$  was NOT held constant, meant that the y-intercept of our trend line was not really constant. Therefore “a” was not just a function of the ratio  $(m_2 - m_1) / (m_1 + m_2)$ , but also  $1 / (m_1 + m_2)$  arising from the friction term.

The percent uncertainty of  $\pm 9.6\%$  in the y-intercept tells us that the friction was non-zero. The non-zero value of the y-intercept shows that there was a measurable amount of friction at work in the experiment. If this percent uncertainty had been close to 100% then it could be argued that the friction was not measurable in our experiment. Don’t make the mistake of saying “the friction was zero” because we know that there is always friction present. The only question is whether it will impact the results of your experiment.

The y-intercept can be shown to be algebraically equal to  $f / (m_1 + m_2)$  if we include the friction in our starting equation as a linear term. Knowing the values of  $m_1$  and  $m_2$  might allow use to estimate the value of the frictional force but in this particular experiment the value of  $m_1 + m_2$  was not kept constant. However, we can take the average value in this case  $(m_1 + m_2)_{\text{Average}} = 116$  grams or 0.116 kg. This will yield a frictional force estimate of

0.00664N and by dividing by “g” (9.80 m/s<sup>2</sup>) we get an equivalent mass of 0.677 grams, which is a little over 1% of the size of the masses we were using in this experiment.

The linear trend analysis, that we will employ, separates the slope from the y-intercept. Therefore we can determine the value of g without the friction causing a problem.

### **Combining the Diameter Uncertainty and the Mass of the Pulley**

$$\% \text{ Slope Error} = \sqrt{(\Delta \text{Diam}\%)^2 + (\Delta \text{PulleyMass}\%)^2}$$

$$\% \text{ Slope Error} = \sqrt{(3.3\%)^2 + (2.5\%)^2}$$

$$\% \text{ Slope Error} = 4.14\%$$

This accounts for much of the 5% error found in the Atwood data.

### **The Solution?**

In this case we have identified a possible “circumference” measurement problem and a massive pulley contribution. There is also a source of variation due to experimental design. A solution is to modify the experiment and require that the sum of the masses remain constant during the experiment. In the version of the lab you will be doing the sum of the masses will remain constant. Will the problem then go away?