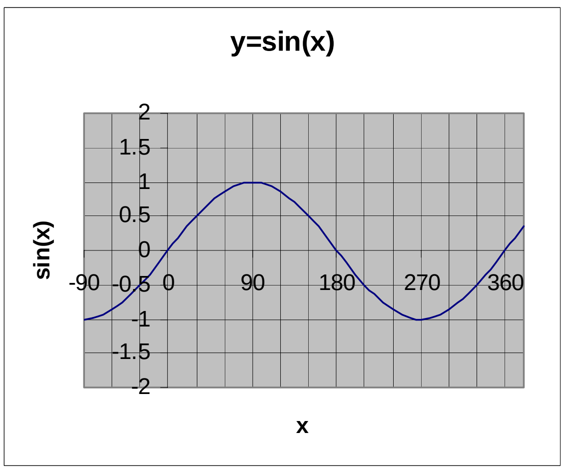
Supplement to Chapter 2, Sections 2 and 3: Solving Trig Equations

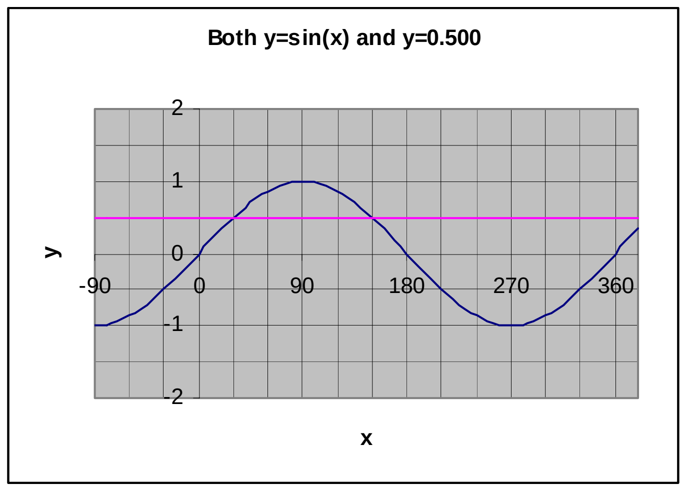
In Chapter 2, section 2, we solved some trig equations whose solutions are special angles using reference angles. See Example 6 in Chapter 2, section 2.

In Chapter 2, section 3, we solved some trig equations whose solutions are just arbitrary angles, not special angles, by using the inverse trig function keys on our calculator. See Example 2 in Chapter 2, section 3.

Those appear to be two different methods of solution of problems that look very similar. Considering appropriate graphs helps us to pull these ideas together.

Consider the trig equation $\sin(x) = 0.5000$ where $0^0 \le x \le 360^0$. Let's use the graph of $y = \sin(x)$ to see how many solutions this equation has over the given domain. See left figure below.





Draw a line $y = 0.5000$ on this graph and let's find the *x*-values at which that line intersects the graph. See the figure on the right above. Do you see that those *x*-values are the solutions to the equation $\sin(x) = 0.5000$?

From the graph we see that the intersection of these graphs occurs at two points on within the given domain: (30,0.5) and (150,0.5) so the solutions to the equation $\sin(x) = 0.500$ are $x = 30^0$ and $x = 150^0$.

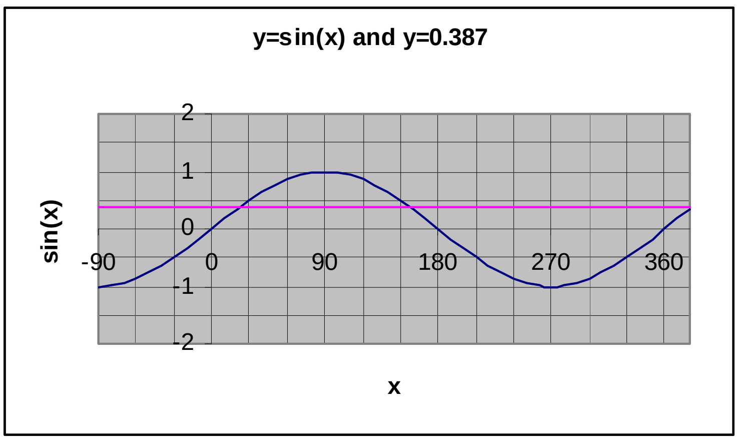
We could have solved the equation $\sin(x) = 0.5000$ where $0^0 \le x \le 360^0$ using the values we had earlier computed and then thinking about reference angles. Earlier we found $\sin(30^0) = \frac{1}{2}$, so $x = 30^0$ is clearly one solution to this equation. Also, since the value of the sine is positive, the unknown angle can be in Quadrant I or II. That means that there is another solution of this equation in Quadrant II. That other solution has reference angle of 30^0 so that angle is $180^0 - 30^0 = 150^0$.

Remember that reference angles are always measured with respect to the *x*-axis, so that means they are either subtracted from or added to either 180^0 or 360^0. Look at the basic trig graphs and notice that each of them has a certain type of symmetry around both 180^0 and 360^0. We can use either the graph or our insight about the signs of the trig functions in the various quadrants to decide which quadrants contain solutions to the equation and how to obtain the additional solutions from the first solution.

How do we check our solutions to problems like this? We can plug in the values we obtained for *x* and see that they solve the equation. That is easy and you should always do at least that much checking. However that does not help us know whether we found all the solutions. Looking at the graph is the best way to determine whether we have found all possible solutions in the given domain.

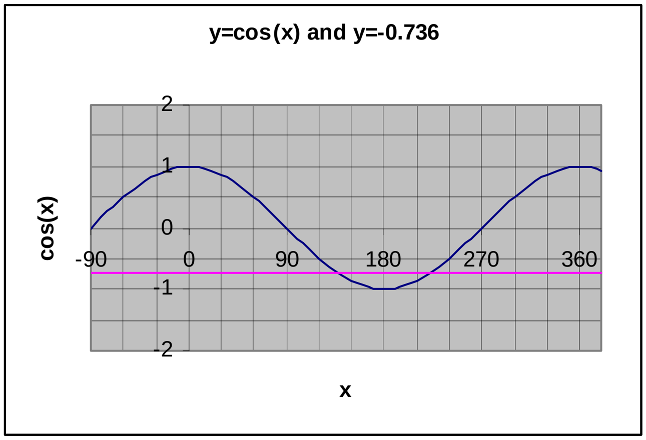
Now let's consider some examples where the intersection points aren't as obvious. We'll also use the inverse trig function on the calculator, as was done in Chapter 2, Section 3.

Example 1: Find all solutions to $\sin(x) = 0.3870$ where $0^0 \le x \le 360^0$.
To solve this, consider the graph



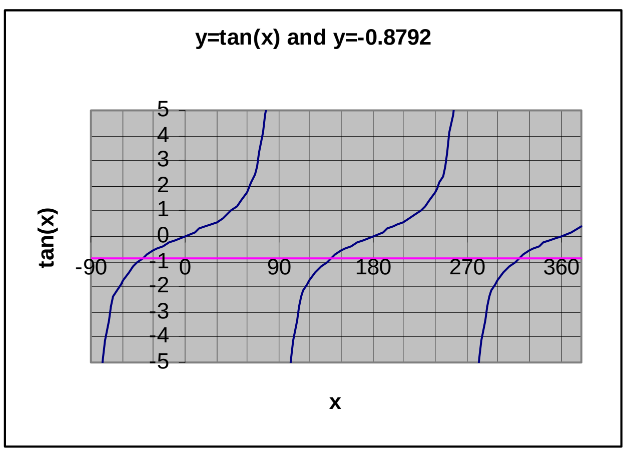
This shows that there is a solution somewhat smaller than 30^0 and another solution somewhat larger than 150^0. To find the first solution exactly, we use a calculator to obtain $\sin^{-1}(0.387) = 22.77^0$. It is clear from the symmetry of the graph (and also from what we know about reference angles) that the other solution is $180^0 - 22.77^0 = 157.23^0$.
Thus, the solution set for $\sin(x) = 0.3870$ where $0^0 \le x \le 360^0$ is $\{22.77^0, 157.23^0\}$.

Example 2: Find all solutions to $\cos(x) = -0.7360$ where $0^0 \le x \le 360^0$.
To solve this, consider the graph



We see two solutions: one between 90^0 and 180^0 and one between 180^0 and 270^0. A calculator gives $\cos^{-1}(-0.736) = 137.39^0$. Obviously that is the first solution and the second solution must be $360^0 - 137.39^0 = 222.61^0$.
Thus, the solution set for $\cos(x) = -0.7360$ where $0^0 \le x \le 360^0$ is $\{137.39^0, 221.61^0\}$.

Example 3: Find all solutions to $\tan(x) = -0.8792$ where $0^0 \le x \le 360^0$.
To solve this, consider the graph



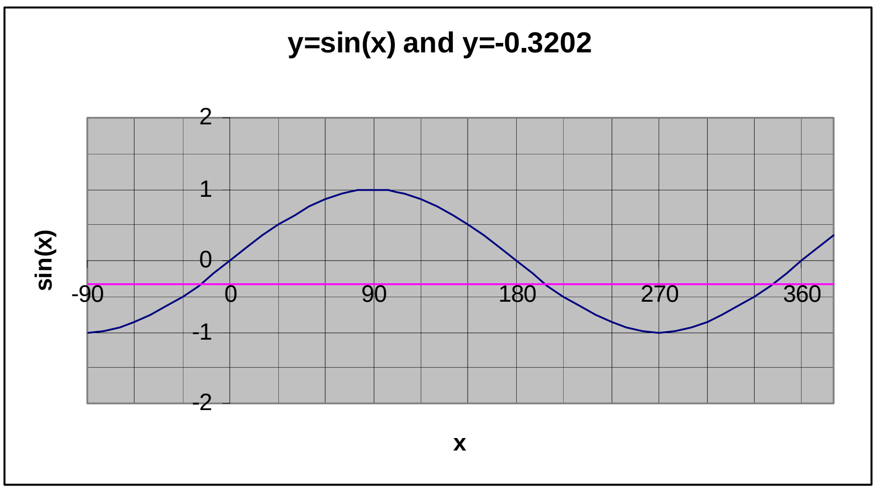
We see three solutions on the graph, but only two in our interval. On the graph, there is one solution between -90^0 and 0^0, another solution between 90^0 and 180^0, and a third solution between 270^0 and 360^0. A calculator gives $\tan^{-1}(-0.8792) = -41.32^0$. That solution is not in our interval, but we can use that to find the two solutions in the interval: $180^0 - 41.32^0 = 138.68^0$ and $360^0 - 41.32^0 = 318.68^0$.
Thus, the solution set for $\tan(x) = -0.8792$ where $0^0 \le x \le 360^0$ is $\{138.68^0, 318.68^0\}$.

Example 4: Find all solutions to $\csc(x) = -3.1235$ where $0^0 \le x \le 360^0$.

To solve this, we could look at the graph of the cosecant function and do it in a similar manner to the previous examples. However, most people would use a reciprocal identity and change this to a problem involving the sine function.

$$\csc(x) = -3.1235$$
$$\frac{1}{\sin(x)} = -3.1235$$
$$\sin(x) = \frac{1}{-3.1235}$$
$$\sin(x) = -0.3202$$

To solve this, we consider this graph:



We see three solutions on the graph, but only two in our interval. On the graph, there is one solution between -90^0 and 0^0, another solution between 180^0 and 270^0, and a third solution between 270^0 and 360^0. A calculator gives $\sin^{-1}(-0.3202) = -18.68^0$. That solution is not in our interval, but we can use that to find the two solutions in the interval. The first one is the same distance above 180^0 as that solution was below 0^0. So that is $180^0 + 18.68^0 = 198.68^0$. The second is the same distance below 360^0, which is $360^0 - 18.68^0 = 341.32^0$.

Thus, the solution set for $\csc(x) = -3.1235$ where $0^0 \le x \le 360^0$ is $\{198.68^0, 341.32^0\}$.

Exercises:

Before doing these problems, create a table of the values for sine, cosine, and tangent for the special angles of 0^0, 30^0, 45^0, 60^0, 90^0, 180^0, 270^0. Don't just copy it – create it yourself in the way you will do it on the test.

Before doing these problems, sketch graphs of the sine, cosine and tangent functions on the domain of $0^0 \le x \le 360^0$. Use those graphs, and a ruler to show the straight line, as needed for finding solutions on this interval.

In 1 - 15, without using a calculator, but using the table **you created**, find the exact value of ϑ, if ϑ is in the interval $\left[0^0, 90^0\right]$, and ϑ has the given function value.

1. $\sin\theta = \frac{\sqrt{2}}{2}$ 2. $\tan\theta = \frac{\sqrt{3}}{3}$ 3. $\cos\theta = \frac{1}{2}$ 4. $\cot\theta = 1$ 5. $\csc\theta = \sqrt{2}$

6. $\sin\theta = \frac{\sqrt{3}}{2}$ 7. $\csc\theta = \frac{2\sqrt{3}}{3}$ 8. $\sec\theta = \sqrt{2}$ 9. $\cot\theta = \sqrt{3}$ 10. $\cos\theta = \frac{\sqrt{3}}{2}$

11. $\sin\theta = 1$ 12. $\sin\theta = 0$ 13. $\tan\theta = 0$ 14. $\cot\theta = 0$ 15. $\cot\theta = 0$

In 16 - 43, without using a calculator, but using the table **you created**, find all exact values of ϑ, if ϑ is in the interval $\left[0^0, 360^0\right]$, and ϑ has the given function value.

16. $\sin\theta = \frac{\sqrt{2}}{2}$ 17. $\tan\theta = \frac{\sqrt{3}}{3}$ 18. $\cos\theta = \frac{1}{2}$ 19. $\cot\theta = 1$ 20. $\csc\theta = \sqrt{2}$

21. $\sin\theta = \frac{\sqrt{3}}{2}$ 22. $\csc\theta = \frac{2\sqrt{3}}{3}$ 23. $\sec\theta = \sqrt{2}$ 24. $\cot\theta = \sqrt{3}$ 25. $\cos\theta = \frac{\sqrt{3}}{2}$

26. $\sin\theta = 1$ 27. $\sin\theta = 0$ 28. $\tan\theta = 0$ 29. $\cot\theta = 0$ 30. $\cot\theta = 1$

31. $\sin\theta = -\frac{\sqrt{2}}{2}$ 32. $\tan\theta = -\frac{\sqrt{3}}{3}$ 33. $\cos\theta = -\frac{1}{2}$ 34. $\cot\theta = -1$ 35. $\csc\theta = -\sqrt{2}$

36. $\sin\theta = -\frac{\sqrt{3}}{2}$ 37. $\csc\theta = -\frac{2\sqrt{3}}{3}$ 38. $\sec\theta = -\sqrt{2}$ 39. $\cot\theta = -\sqrt{3}$ 40. $\cos\theta = -\frac{\sqrt{3}}{2}$

41. $\sin\theta = -1$ 42. $\csc\theta = -1$ 43. $\sec\theta = -1$

In problems 44-58, find all solutions of each equation in $0^0 \le x \le 360^0$, correct to the nearest hundredth of a degree.

44. $\cos(x) = 0.35$ 45. $\sin(x) = 0.16$ 46. $\sin(x) = -0.75$ 47. $\cos(x) = -0.83$ 48. $\tan(x) = 2.4$

49. $\cos(x) = 0.65$ 50. $\sin(x) = 0.82$ 51. $\sin(x) = -0.14$ 52. $\cos(x) = -0.35$ 53. $\tan(x) = -0.54$

54. $\sec(x) = 1.84$ 55. $\csc(x) = -4.8$ 56. $\cot(x) = 0.84$ 57. $\sec(x) = 2.23$ 58. $\cot(x) = -2.7$

Answer Key:

1. $\{45^0\}$ **2.** $\{30^0\}$ **3.** $\{60^0\}$ **4.** $\{45^0\}$ **5.** $\{45^0\}$

6. $\{60^0\}$ **7.** $\{60^0\}$ **8.** $\{45^0\}$ **9.** $\{30^0\}$ **10.** $\{30^0\}$

11. $\{90^0\}$ **12.** $\{0^0\}$ **13.** $\{0^0\}$ **14.** $\{90^0\}$ **15.** $\{0^0\}$

16. $\{45^0, 135^0\}$ **17.** $\{30^0, 210^0\}$ **18.** $\{60^0, 300^0\}$ **19.** $\{45^0, 225^0\}$ **20.** $\{45^0, 135^0\}$

21. $\{60^0, 120^0\}$ **22.** $\{60^0, 120^0\}$ **23.** $\{45^0, 315^0\}$ **24.** $\{30^0, 210^0\}$ **25.** $\{30^0, 330^0\}$

26. $\{90^0\}$ **27.** $\{0^0, 180^0, 360^0\}$ **28.** $\{0^0, 180^0, 360^0\}$ **29.** $\{90^0, 270^0\}$ **30.** $\{0^0, 360^0\}$

31. $\{225^0, 315^0\}$ **32.** $\{150^0, 330^0\}$ **33.** $\{120^0, 240^0\}$ **34.** $\{135^0, 315^0\}$ **35.** $\{225^0, 315^0\}$

36. $\{240^0, 300^0\}$ **37.** $\{240^0, 300^0\}$ **38.** $\{135^0, 225^0\}$ **39.** $\{150^0, 330^0\}$ **40.** $\{150^0, 210^0\}$

41. $\{270^0\}$ **42.** $\{270^0\}$ **43.** $\{180^0\}$

44. $\{69.51^0, 290.49^0\}$ **45**. $\{9.21^0, 170.79^0\}$ **46**. $\{48.59^0, 311.41^0\}$ **47**. $\{146.10^0, 213.90^0\}$

48. $\{67.38^0, 247.38^0\}$ **49**. $\{49.46^0, 310.54^0\}$ **50**. $\{55.08^0, 124.92^0\}$ **51**. $\{188.05^0, 351.95^0\}$

52. $\{110.49^0, 249.51^0\}$ **53**. $\{151.63^0, 331.63^0\}$ **54**. $\{57.08^0, 302.92^0\}$ **55**. $\{192.02^0, 347.98^0\}$

56. $\{49.97^0, 229.97^0\}$ **57**. $\{63.36^0, 296.64^0\}$ **58**. $\{159.68^0, 339.68^0\}$