Topic D. Formulas – Computing and Graphing

1. Distinguish between input values and output values in formulas.
2. Use the order of operations correctly evaluating formulas.
3. Choose an appropriate scale for each axis and graph the formula by hand.
4. Use the graph to find what input value will give a particular output value.
5. Use numerical methods to check that answer and to refine it to get a somewhat more accurate answer. (After learning to graph using a spreadsheet.)
6. Use formulas with several different variables as input.
7. Understand the use of subscripts in formulas.

Many formulas are used in geometry and other applications. It is important to be able to correctly plug in values and compute the result. In this course we will also learn to graph some of these formulas so that we can investigate the patterns more fully.

DEFINITION: Usually when people refer to an equation as a formula it has the output variable alone on one side of the equation. When the equation is written in that form, it is easy to evaluate the output variable at different input values and to easily graph the formula. **When we use the word "formula" in this course, we will always mean that the output variable is alone on one side of the equation**.

Example 1. In Canada and Mexico, weather reports report temperature using the Centigrade (or Celsius) scale. In the US, temperature is reported using the Fahrenheit scale. To convert temperature C to temperature F, we use the formula $F = \frac{9}{5}C + 32$.

When the temperature C is 23°, what is the temperature F?

Solution to Example 1:

$$F = \frac{9}{5}C + 32$$

$$F = \frac{9}{5}(23) + 32$$

$$F = 41.4 + 32$$

$$F = 73.4$$

Thus the temperature F is 73.4°.

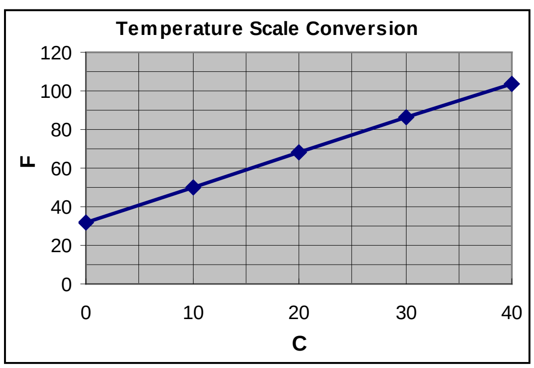
Notice that we used the correct order of operations here, performing the multiplication before the addition.

Here we call the 23° the input value and the 73.4° the output value. (In algebra class, we call the set of input values the "domain" and the set of output values the "range".)

When we graph a formula, we will put the input value on the horizontal axis, sometimes called the *x*-axis, and the output value on the vertical axis, sometimes called the *y*-axis. We can use different letters for the variables besides *x* and *y*, but when we think of the formula graphically, we must be clear about which variable plays the role of *x* and which plays the role of *y*. (You should be able to do these graphs by hand as well as with a spreadsheet.)

Formula input and output

C	F
23	73.4
0	32
10	50
20	68
30	86
40	104



Example 2. If we invest \$750 at 6% annual interest, compounded quarterly, the formula for the amount A that the investment is worth after t years is $A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4t}$. Find the amount the investment is worth after five years. Then evaluate this for several values of t between 0 and 30 years, and sketch a graph of this formula.

Solution: First, recall how to get an exponent in your calculator. On most calculators, it's a ^ key or a y^x key. Find that key and practice using it to evaluate 2^3. When you can do that correctly, then evaluate the following expression, writing these intermediate values as indicated below.

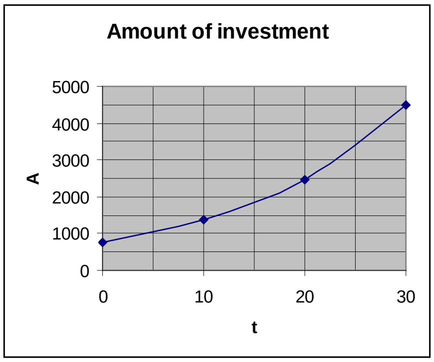
$$A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4t} = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4 \cdot 5} = 750 \cdot (1.015)^{20} = 750 \cdot 1.346855007 = 1010.141255$$

So the investment is worth \$1010.14 after 5 years.

It is more convenient to be able to put all of these values into the calculator at the same time rather than writing intermediate steps. We will learn to do that in this course, but that's not the point of this lesson. The point here is to practice making graphs. This will be useful to you in checking your work when you evaluate formulas. If one of the values you compute doesn't seem to fit the pattern of the others, that warns you to check your computation again.

Formula input and output

t	A
5	1010.141
0	750.000
10	1360.514
20	2467.997
30	4476.992



We can use the graph to estimate at what time the amount of the investment will be about \$4000. Look at the graph to find 4000 on the vertical axis and then notice that the corresponding value on the horizontal axis is about 27.5. So at approximately 27.5 years, the amount of the investment will be \$4000.

Check that by plugging it into the formula. $A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4 \cdot 27.5} = \text{ etc. } = 3857.68$

This is as close as we could reasonably expect from using a graph to approximate the input value.

Example 3. Consider our approximation from Example 2. We wanted to find the number of years to leave the money in so that the amount of the investment would be \$4000. The graph suggested that $t = 27.5$ years. But then we found that after 27.5 years, the amount was only \$3857.68. Clearly we should leave the money in somewhat longer. How much longer?

If we know how to do some complicated algebra (using logarithms) we can quickly obtain an answer to "how long should we leave the money to earn \$4000?" We will not do that particular type of algebra in this course. Instead we will use numerical work to get refine our estimate from the graph. Here, when we look at the graph and the numbers, it is clear that 30 years is too long and 27.5 years is too short. So we

try some value in between. Let's try 28 years. $A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4 \cdot 28} = \text{etc.} = 3974.28$. This is much closer to the $4000 which was our goal. If we wanted to try to get it even more accurately, we would use a value larger than 28 years, but only very slightly larger. Since the amount is compounded quarterly, the next time the amount will increase is at 28.25 years, so let's try that.
$A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4 \cdot 28.25} = 4033.89$. That is further away from $4000 than the value for 28 years, so the best answer for this question is that the amount of the investment will be at about $4000 at 28 years.

Example 4. Following example 2, we'd like to have a formula that allows us to vary the interest rate and number of times per year it is compounded as well as varying the number of years of the investment. So let *r* be the interest rate, converted to a decimal and *n* be the number of times per year it is compounded, and, as before *t* is the number of years and *A* is the amount of the investment. Then the formula is
$A = 750 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$. This is an example of a formula with several input values. Use this formula to find the amount of the investment after 5 years if the interest rate is 8% and it is compounded monthly.
Solution:

$$A = 750 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t} = 750 \cdot \left(1 + \frac{0.08}{12}\right)^{12 \cdot 5} = 750 \cdot (1.006666667)^{60} = 750 \cdot 1.4898457 = 1117.384281$$

So after 5 years, this investment is worth $1117.38.

We can't graph this easily, because we would need more than a two-dimensional graph. We need one dimension for the output value and one dimension for every input value and investigating four-dimensional graphs is beyond the scope of this course! However, in most practical applications, technicians and scientists isolate one or two input variables that they are most interested in and then analyze the problem with a two-dimensional graph (for one input variable) or a three-dimensional graph (for two input variables.) We will look at some three-dimensional graphs later in the course.

Example 5. Following Example 3, suppose we want to allow the initial amount of the investment to change. So we need a variable for that. Since it is an amount, we'd like to call it *A*, but we already have an *A* in this formula that means something else. We could use a different letter, but in applications problems we often choose to call both values *A* and distinguish between them by a subscript. In this problem, we would usually call the original amount of money A_0 and the final amount of money after *t* years A_t. So the formula is $A_t = A_0 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$. Use this formula to find the amount of an investment after 6 years if the initial amount is $900, the annual rate is 0.07 and it is compounded twice a year.

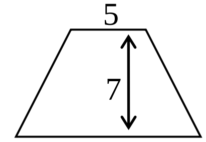
Solution:

$$A_t = A_0 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

$$A_6 = 900 \cdot \left(1 + \frac{0.07}{2}\right)^{2 \cdot 6} = 900 \cdot (1.035)^{12} = 900 \cdot 1.511068657 = 1359.961792$$

So, after 6 years, this investment is worth $1359.96.

Example 6. Find the area of this trapezoid, where the numbers represent feet.



The formula is $A = \frac{1}{2}h\left(b_1 + b_2\right)$ where h is the height, b_1 and b_2 are the lengths of the two bases, and A is the area. Notice that the subscripts are used because both these sides are called bases, so b is a reasonable letter to use, but there are two different ones. We use subscripts on the b's to distinguish between the two different bases.

$$A = \frac{1}{2}h\left(b_1 + b_2\right) = \frac{1}{2}\cdot 7\left(5+9\right) = 49 \ \ ft^2$$

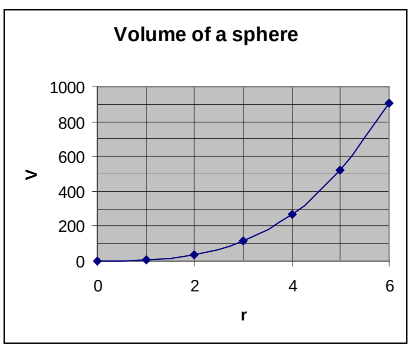
Example 7. Find the volume of a sphere with radius 3 inches. The formula is $V = \frac{4}{3}\pi r^3$.

Solution: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3\right)^3 = \frac{4}{3}\pi\cdot 27 = 113.097$.
So the volume of this sphere is 113.097 cubic inches.

Example 8. Graph the formula $V = \frac{4}{3}\pi r^3$ for the values of r from 0 to 6 feet and use the graph to approximate what radius will give a volume of 400 cubic feet

Solution. Notice that r is the input value and V is the output value.

r	V
0	0
1	4.188787
2	33.51029
3	113.0972
4	268.0823
5	523.5983
6	904.7779



Using the graph, we follow the line for 400 on the vertical axis across to the graph and then down to see the corresponding r value, which is approximately $r = 4.5$ feet. We can check this by plugging in and finding that $V = \frac{4}{3}\pi(4.5)^3 = 381.7$

If we wanted to get a better estimate, we would try a somewhat higher value for r.

Example 9. Consider the problem of Example 8. Use graphical and numerical methods to find a value for the radius that will give a volume of 400 cubic feet, correct to within 10 cubic feet.

Solution: The graph tells us that a good estimate for the radius is 4.5 feet. Next we check that by plugging in and finding that $V = \frac{4}{3}\pi(4.5)^3 = 381.7$ cubic feet. That's a bit too low, so let's try a slightly larger value for the radius. Let's try 4.6 feet. $V = \frac{4}{3}\pi(4.6)^3 = 407.72$ cubic feet. That's an adequate answer, according to the tolerance level stated in the problem. Of course, if we wanted a more accurate answer, we could continue to work numerically by taking a value for the radius just a bit smaller than 4.6 feet and finding the volume for that.

Perhaps you have noticed that you could use algebra (involving cube roots) instead of the graph on the problem of Examples 8 and 9 to obtain a quite exact answer for the radius needed to obtain a volume of 400 cubic feet. And that's a fine thing to do. But, in this course, we are practicing this technique that will allow you to solve problems of this kind even if the formula is so complicated that the algebra needed to "invert" the formula is harder than anything you've learned in your algebra courses. In fact, some formulas are so complicated that even mathematicians find these values in just the same way you're learning here. Using graphs in this way is a very powerful and important tool in working on technical problems!

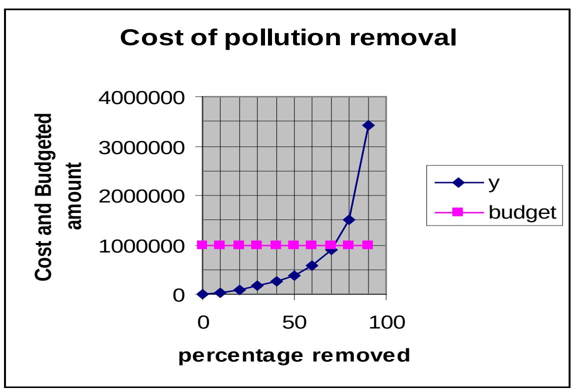
Example 10. A certain river in Georgia is polluted with coliform bacteria. This has been blamed primarily on the dairy farmers whose farms are in the watershed of the river. This formula is used to model the cost in dollars, *y*, to remove *p*% of the bacterial pollution from the river. How much of the pollution can be removed if $1.0 million is available to spend on this? $y = \frac{380,000p}{100 - p}$

Solution. We must graph this formula over the range of values $0 \le p < 100$ and then see for what values of *p* the *y* value is less than or equal to $1.0 million.

Use parentheses when putting this formula into your calculator. When we write this by hand it is obvious that the entire numerator and entire denominator should be evaluated separately before dividing. But when we enter it into a calculator, we have to use parentheses to indicate that.

For *p* = 20, use (380000*20)/(100 – 20) =

p	y
0	0
10	42222.22
20	95000.00
30	162857.1
40	253333.3
50	380000.0
60	570000.0
70	886666.7
80	1520000.0
90	3420000.0



We graphed the formula and then put a line across at $1 million. This crosses the cost graph at a bit more than $x = 70$. So we estimate that about 72% of the pollution can be removed for $1.0 million.

We could expand the scale or do more numerical computation to estimate this more precisely. That would be tedious to do by hand, but would be easy to do with a spreadsheet.

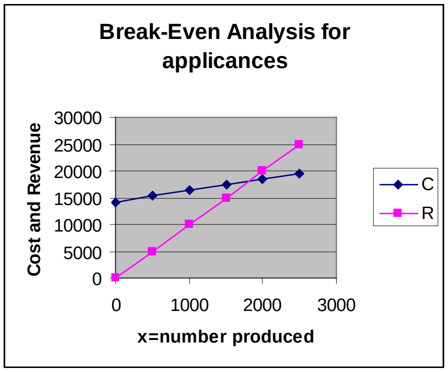
Example 11. A certain kind of appliance is sold for $10 each and the manufacturer can sell all she produces at that price. To produce these appliances requires a fixed cost of $14,220 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per appliance is $2.10.

a. Write a formula for the cost of producing *x* appliances.
b. Write another formula for the revenue produced by selling *x* appliances.
c. Graph both formulas for $0 \le x \le 2500$.
d. For what value of *x* is the cost equal to the revenue? (That point on the graph is called the "break-even" point.)

Solution:

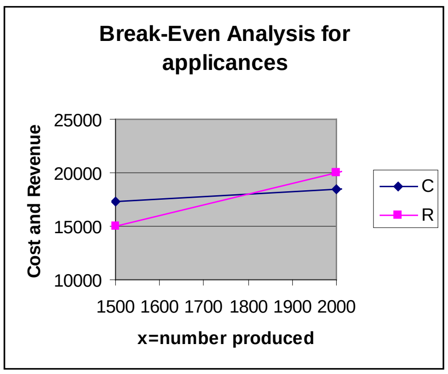
a. Let C = total cost of *x* appliances. C = fixed cost + variable cost times number of appliances.
Thus $C = 14220 + 2.10 \cdot x$

Let R = total revenue from *x* appliances. R = price per appliance times number of appliances
Thus $R = 10.00 \cdot x$



x	C	R
0	14220	0
500	15270	5000
1000	16320	10000
1500	17370	15000
2000	18420	20000
2500	19470	25000

Notice that these are equal for some x-value about halfway between 1500 and 2000. So we plot more values in this range and find that they are equal at *x* = 1800.



We plot more values in this range and find that the cost and the revenue are equal at *x* = 1800.

So, for 1800 appliances, both the cost and the revenue are \$18,000. That means that, if we make 1800 appliances, we will "break even."

Exercises: Use a calculator and make the graphs by hand to work the following problems. Don't calculate very many points – use those given. In this class we will usually use a spreadsheet to automate the calculations. The work in this lesson is mainly to help you recall how to graph formulas by hand so that you will fully understand the graphs you produce with a spreadsheet.

Part I.

1. If we invest \$750 at 6% annual interest, compounded quarterly, the formula for the amount *A* that the investment is worth after *t* years is $A = 750 \cdot \left(1 + \frac{0.06}{4}\right)^{4t}$.
    a. Find the amount the investment is worth after five years. Then evaluate this for two more values of *t* between 0 and 30 years. Use the rest of the values that were given in the example, and sketch a graph of this formula.
    b. For the previous problem, how long should we leave the investment so that it will be worth \$4000? Estimate the time, correct to the nearest year.
2. Then the formula is $A = 750 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$. This is an example of a formula with several input values. Use this formula to find the amount of the investment after 5 years if the interest rate is 8% and it is compounded monthly.

3. Find the area of a trapezoid with bases 5 feet and 9 feet and height 7 feet. $A = \frac{1}{2}h\left(b_1 + b_2\right)$

4. Graph the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, for the values of r from 0 to 6 feet and use the graph to approximate what radius will give a volume of 400 cubic feet, correct to within 10 cubic feet. Evaluate about three values by hand/calculator and then use the rest from the example.

5. A certain river in Georgia is polluted with coliform bacteria. This has been blamed primarily on the dairy farmers whose farms are in the watershed of the river. This formula is used to model the cost in dollars, *y*, to remove *p*% of the bacterial pollution from the river. How much of the pollution can be removed if $1.0 million is available to spend on this? Evaluate about three values by hand/calculator and then use the rest form the example. $y = \frac{380{,}000p}{100 - p}$

6. A certain kind of appliance is sold for $10 each and the manufacturer can sell all she produces at that price. To produce these appliances requires a fixed cost of $14,220 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per appliance is $2.10.
   a. Write a formula for the cost of producing *x* appliances.
   b. Write another formula for the revenue produced by selling *x* appliances.
   c. Graph both formulas for $0 \le x \le 2500$.
   d. For what value of *x* is the cost equal to the revenue? (That is called the "break-even" point.

Part II.

7. The concentration of a particular drug in a patient's blood is given by the formula $C = \frac{7}{0.3t^2 + 1.1}$ where *t* is the number of hours after the patient takes the medicine and *C* is the concentration in milliliters per liter of blood. Of course, $t \ge 0$. (Hint: When you are using your calculator to evaluate this, be sure to put parentheses around the entire denominator of the fraction.)

t	*C*
0	6.363636
1	5.000000
2	3.043478
3	
4	1.186441
5	0.813953

   a. Use your calculator to confirm at least one of the values of *C* in the table of values and then fill in the missing value.
   b. Which is the input variable? (When you graph this, put that along the horizontal axis.)
   c. Using graph paper, graph this formula by hand.
   d. It is recommended that the patient not take another dose until the concentration is less than 2.0 ml/l. According to your graph of the formula, approximately how long will that take? (Estimate to the nearest hour.)

8. The landing speed S, in feet per second, of a particular type of small plane can be modeled by the formula $S = \sqrt{1.496w}$ where w is the weight in pounds of the plane.

w	S
4000	77.35632
6000	94.74175
8000	
10000	122.3111

a. Use your calculator to confirm at least one of the values of C in the table of values and then fill in the missing value..
b. Which is the input variable? (When you graph this, put that along the horizontal axis.)
c. Using graph paper, graph this formula by hand.
d. What weight would correspond to a landing speed of 100 ft/sec? Use your graph to estimate this to the nearest thousand pounds.

9. The logistic formula can be used to describe population growth in a situation with limited resources. Consider this logistic formula $P = \dfrac{1000}{1+10\cdot(2^{-t})}$, where P is the population at time t.

(Hint: When you are using your calculator to evaluate this, be sure to put parentheses around the entire denominator of the fraction.)

t	P(t)
0	90.9091
1	166.6667
2	285.7143
3	444.4444
4	
5	761.9048
6	864.8649
7	
8	962.4060
9	980.8429

a. Use your calculator to confirm at least one of the values of P in the table of values and then fill in the missing value.
b. Which is the input variable? (When you graph this, put that along the horizontal axis.)
c. Using graph paper, graph this formula by hand.
d. What time would we expect to have a value for P of 900? Use your graph to estimate this.

10. A certain kind of thermometer is sold for $5 each and the manufacturer can sell all she produces at that price. To produce these thermometers requires a fixed cost of $3500 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per thermometer is $1.50.
    a. Write a formula for the cost of producing x thermometers.
    b. Write a formula for the revenue produced by selling x thermometers.
    c. Graph both formulas for $0 \le x \le 2500$ in increments of 500.
    d. For what value of x is the cost equal to the revenue? (That point on the graph is called the “break-even” point.)
    e. What is the total cost of making that many thermometers?
    f. What is the total revenue earned from that many thermometers?