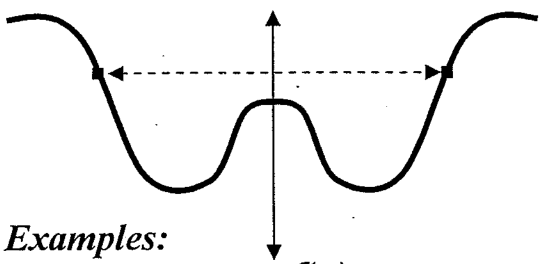
Even Functions $\mathcal{E}$ and Odd Functions $\mathcal{O}$

Even Symmetry of f

f even if for all x, $f(-x) = f(x)$

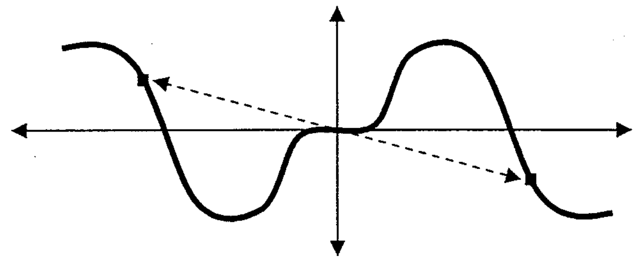
Graph symmetry across the line x = 0 (= y axis)



Odd Symmetry of g

g odd if for all x, $g(-x) = -g(x)$

Graph symmetry across the point (x,y) = (0,0)



Examples:

Constant functions: $f(x) = 3, \ -0.77, \ \text{pi}$

$f(x) = \cos(x), \ \sec(x)$

Even integer powers: $f(x) = x^2, \ x^6, \ x^{-50}$

Even numerator fractional powers: $f(x) = x^{2/3}, \ x^{-14/9}$

$g(x) = \sin(x), \ \tan(x), \ \csc(x), \ \cot(x)$

Odd integer powers: $g(x) = x, \ x^3, \ x^{-5}$

Odd numerator fractional powers: $g(x) = x^{3/5}, \ x^{-7/3}$

Note: In both of the above fractional powers, the denominator must be odd. Even denominator fractional powers like $\sqrt{x} = x^{1/2}$ are not defined for x < 0 and so are not symmetric.

Theorems for combinations of functions:

Where $\mathcal{E}$ represents ANY even function(s) and $\mathcal{O}$ represents ANY odd function(s) which may vary.

$\mathcal{E} + \mathcal{E} = \mathcal{E}$ $\quad \mathcal{E} - \mathcal{E} = \mathcal{E}$ $\quad$ **$x^2 + 3$ is even** $\quad$ **$\cos(x) - x^{-2}$ is even**

$\mathcal{O} + \mathcal{O} = \mathcal{O}$ $\quad \mathcal{O} - \mathcal{O} = \mathcal{O}$ $\quad$ **$x^3 + 3/x$ is odd** $\quad$ **$\tan(x) - 4x^{5/3}$ is odd**

$\boxed{\mathcal{E} \pm \mathcal{O} \text{ and } \mathcal{O} \pm \mathcal{E} \text{ are NOT SYMMETRIC}}$ **$3 + x^3$ & $\tan(x) + \cos(x)$ asymmetric**

$\mathcal{E} * \mathcal{E} = \mathcal{E}$ $\quad \mathcal{E} / \mathcal{E} = \mathcal{E}$ $\quad$ **$(x^2 + 3)\cos(x)$ is even** $\quad$ **$(x^2 + 3) / x^2$ is even**

$\mathcal{O} * \mathcal{O} = \mathcal{E}$ $\quad \mathcal{O} / \mathcal{O} = \mathcal{E}$ $\quad$ **$(x^3 + 3x) / x^{3/5}$ is even** $\quad$ **$4x^{5/3}\tan(x)$ is even**

$\mathcal{E} * \mathcal{O} = \mathcal{O} * \mathcal{E} = \mathcal{E} / \mathcal{O} = \mathcal{O} / \mathcal{E} = \mathcal{O}$ $\quad$ **$(x^2 + 3)(x^3 + 4x^{-5})$ is odd.**

So, multiplying by any coefficient or taking reciprocals does NOT change symmetry:

$11.2(x^5 - 4x)$ **and** $\dfrac{\sqrt{\pi}}{x^5 - 4x}$ **are still odd.** $\quad 7x^2$, $\dfrac{3}{x^2 - \pi}$ **and** $\dfrac{3e}{\cos(x) - 7x^2}$ **are still even.**

Composition: $\mathcal{O} \circ \mathcal{O} = \mathcal{O}$ $\quad$ **$(x^3 + 3/x)^5$ is odd** $\quad$ **$\tan(4x^{5/3})$ is odd**

$\mathcal{E} \circ \mathcal{O} = \mathcal{E}$ $\quad$ ***(ANY fn)*** $\circ\ \mathcal{E} = \mathcal{E}$ $\quad$ **$(x^3 + 3/x)^{-4}$ is even** $\quad$ **$(x^2 + 3)^{1/2} - \tan(x^2 + 3)$ is even**

Proof of $\mathcal{O} + \mathcal{O} = \mathcal{O}$: Let f and g be odd functions. Then

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x)$$

Proof of $\mathcal{E} / \mathcal{E} = \mathcal{E}$: Let f and g be even functions. Then

$$(f/g)(-x) = f(-x)/g(-x) = f(x)/g(x) = (f/g)(x)$$

Proof of *(ANY fn)* $\circ\ \mathcal{E} = \mathcal{E}$: Let f be any function and g be even. Then

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$$

✓ $\sqrt{(x^2+2)^3 - 1}$