Chapter 8 Summary Sheet

Series

1. The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \text{ when } |r| < 1$$

When $|r| \geq 1$ the series diverges.

2. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

3. The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent otherwise.

Theorems, Tests and Definitions

Section 8.1

1. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$
2. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
3. Every bounded, monotonic sequence is convergent.

Section 8.2

1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
2. **Test for Divergence**

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Section 8.3

1. **Integral Test**

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

2. **The Comparison Test**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

a) If $\sum b_n$ is convergent and $a_n \le b_n$, then $\sum a_n$ is convergent.

b) If $\sum b_n$ is divergent and $a_n \ge b_n$, then $\sum a_n$ is divergent.

3. **The Limit comparison Test**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

4. **Remainder Estimate for the Integral Test**

If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

Section 8.4

1. **The Alternating Series Test**

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ with $b_n > 0$ satisfying

a) $b_{n+1} \le b_n$

b) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Note: The harmonic series $\sum \frac{1}{n}$ is divergent but the alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ converges.

2. **Alternating Series Estimation Theorem**

If $s = \sum (-1)^{n-1} b_n$ is the sum of the alternating series that satisfies

a) $b_{n+1} \le b_n$ and b) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

3. **Absolute Convergent** A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

4. **The Ratio Test**

   a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent.)

   b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ for $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Section 8.5

1. A **power series in *x*** (or a power series centered at 0 or a power series about 0) converges and has the form

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

2. A **power series in (*x* – *a*)** (or a power series centered at *a* or a power series about *a*) has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

3. **Sum of a power series in (*x*- *a*) possibilities**

   For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

   i) The series converges only when $x = a$.
   ii) The series converges $\forall x$.
   iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

   R is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

Section 8.6

1. The power series $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, then the function *f* defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

   i. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} c_n (x-a)^{n-1}$

   ii. $\int f(x)\,dx = C + c_0 + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum c_n \frac{(x-a)^{n+1}}{n+1}$

   The radii of convergence of the power series in Equations (i) and (ii) are both R.

2. If the power series $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

i. $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^3 + \cdots = \sum_{n=1}^{\infty} c_n (x-a)^{n-1}$$

ii. $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Note 1: When dealing with power series, equation (i) and (ii) can be rewritten as

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$

The derivative of a sum is the same as the sum of the derivatives.

(iv) $$\int\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx$$

The integral of a sum is the same as the sum of the integrals.

Note 2:
Although the radius of convergence remains the same when a power series is differentiated or integrated, the *interval of convergence* may change. Any change will be at an endpoint.

Note 3:
The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations.