Basic Function Concepts

Function:
Describes relationship between input variable and output value.

Terminology:
* Input variable = function variable; variable; independent variable
  Common notation for function variable is: $x$
* Output value = function value; dependent variable
  Common notation for function value is: $f(x)$, or $y$
* Domain = set of all valid (legal) inputs
* Range = set of output values generated by all valid inputs

Fundamental Rule:
Functions must always, always, always be single-valued!
Every input value (or, point in the domain) must lead to exactly one output value.
You cannot, for example, have $f(2)=-1$ and $f(2)=6$ in the same function.
On the other hand, it is okay to get the same output from different input values: $f(2)=-1$ and $f(6)=-1$. 

Which of these are functions?
Concept of Limit

* The limit of f(x) at x=c is "sort of like" the function value at x=c, i.e., f(c).

* But ... this similarity is delusional!

* Conceptually, the limit of f(x) at x=c is unrelated to f(c).

* Best way to understand limits is to look at graph of f(x).

* Recipe to find the limit at x=c:

  1. "Get on" the graph somewhere near the point x=c.
  2. Head towards x=c, and note the y-value when you reach x=c.
  3. Repeat steps (1) and (2) by approaching from the opposite side.
  4. If you get the same y-value in steps (2) and (3), this is the limit.

* Example:

  **Case 1:** At x=2, the function value and limit value are the same:
  
  \[ f(2) = \lim_{x \to 2} f(x) = 4. \]

  **Case 2:** At x=2, function value and limit value not the same:

  \[ f(2) = \text{DNE}, \quad \lim_{x \to 2} f(x) = 4. \]

Q: Now, compare function value and limit value at some other point, say, x=1.
<table>
<thead>
<tr>
<th>Function value at x=c</th>
<th>Limit value of f(x) as x approaches c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only depends on f(c).</td>
<td>Does not depend upon f(c).</td>
</tr>
<tr>
<td>Does not depend upon neighboring values.</td>
<td>Only depends on neighboring values.</td>
</tr>
<tr>
<td>May exist even if ( \lim_{x \to c} f(x) ) does not exist.</td>
<td>May exist even if f(c) does not exist.</td>
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</tbody>
</table>

**Left limit and right limit**

* Left limit: Only look at graph to the left of the point of interest (x=c). Ignore graph to the right.

  "Get on" the graph & approach x=c from the left.

  The y-value when you reach x=c is the left limit.

* Right limit: Only look at graph to the right of the point of interest (x=c). Ignore graph to the left.

  "Get on" the graph & approach x=c from the right.

  The y-value when you reach x=c is the right limit.

* Revisit previous example, Case2, at x=2:

  Left limit (LL): \( \lim_{x \to 2^-} f(x) = 4 \)

  Right limit (RL): \( \lim_{x \to 2^+} f(x) = 4 \)

  ***** Q: Are the LL and RL the same always? *****
Example: (Postal rates)

The rate $R$ of first class postage (in cents) is a function of the weight $w$ (in ounces), and is given by:

$$R(w) = \begin{cases} 0.37, & \text{if } 0 < w \leq 1 \\ 0.60, & \text{if } 1 < w \leq 2 \\ 0.83, & \text{if } 2 < w \leq 3 \end{cases}$$

Find $\lim_{w \to 1^-} R(w)$, $\lim_{w \to 1^+} R(w)$, $\lim_{w \to 1^-} R(w)$, $\lim_{w \to 1^+} R(w)$.

Find $\lim_{w \to 1.5^-} R(w)$, $\lim_{w \to 1.5^+} R(w)$, $\lim_{w \to 1.5^-} R(w)$, $\lim_{w \to 1.5^+} R(w)$.

Example 2: Consider the function $g(t)$ defined in the domain $-1 \leq t < \infty$:

$$g(t) = \begin{cases} -\sqrt{1-t^2}, & \text{if } -1 \leq t < +1 \\ +1, & \text{if } t = 1 \\ t-1, & \text{if } 1 < t \leq 2 \\ 0, & \text{if } t > 2 \end{cases}$$

Consider three separate points of the form $t=c$, with $c=0$, $c=1$, $c=2$. For each point, answer the following questions:

(1) What is $g(c)$?

(2) What is $\lim_{t \to c^-} g(t)$, $\lim_{t \to c^+} g(t)$, $\lim_{t \to c^-} g(t)$, $\lim_{t \to c^+} g(t)$?

(3) Is the function continuous at $t=c$?
Example: Find \( \lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} \).

**Before algebra**

\[ y = \frac{x^2 + x - 6}{x + 3} \]

**After algebra**

\[ y = x - 2 \]

Example 2: Find \( \lim_{x \to 1} \left(1 + \frac{1}{x-2}\right) \left(\frac{2}{1-x^2}\right) \).

**Before algebra**

\[ y = \left(1 + \frac{1}{x-2}\right) \left(\frac{2}{1-x^2}\right) \]

**After algebra**

\[ y = \frac{-2}{(x-2)(1+x)} \]
Example: Find \( \lim_{{x \to 0}} \frac{x - 2}{{x^2 - 2x}} \).

**Before algebra**

\[ y = \left( \frac{x - 2}{{x^2 - 2x}} \right) \]

**After algebra**

\[ y = \left( \frac{1}{x} \right) \]

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Example: Find \( \lim_{{t \to 1}} \frac{\sqrt{t+3} - 2}{{t - 1}} \).

**Before algebra**

\[ g(t) = \left( \frac{\sqrt{t+3} - 2}{{t - 1}} \right) \]

**After algebra**

\[ g(t) = \left( \frac{1}{{\sqrt{t+3} + 2}} \right) \]