Inverse Functions

If \( f(g(x)) = x \), for all values of \( x \) in the domain of \( g \), and \( g(f(x)) = x \), for all values of \( x \) in the domain of \( f \), then \( f \) and \( g \) are called **inverse functions**. In other words, if two functions are inverses, then when the output from either function is used as the input for the other function, the result is the same as if nothing had been done to the original input. For example, if \( f(2) = 7 \) and \( g(7) = 2 \), then \( f(g(7)) = f(2) = 7 \) and \( g(f(2)) = g(7) = 2 \). The inverse function of \( f(x) \) is usually denoted \( f^{-1}(x) \).

Notice that if \((2,7)\) is an ordered pair in the function \( f \), then \((7,2)\) must be an ordered pair in its inverse function. In other words, the \( x \)'s and \( y \)'s of the ordered pairs of the function \( f(x) \) are interchanged in the ordered pairs of the inverse function \( f^{-1}(x) \). So the domain of \( f(x) \) is the range of \( f^{-1}(x) \) and the range of \( f(x) \) is the domain of \( f^{-1}(x) \). For example, the domain of \( f(x) = e^x \) is \((-\infty, \infty)\), so \((-\infty, \infty)\) is the range of the inverse of this function. Similarly, the range of \( f(x) = e^x \) is \((0, \infty)\), so \((0, \infty)\) is the domain of the inverse of this function.

If any \( y \)-values in a function are repeated as \( y \)-values, then when the coordinates are interchanged to produce the inverse function, repeated \( x \)-values result and the inverse would not be a function. A function that has no repeated \( y \)-values is called a **one-to-one function** because each \( x \)-value has exactly one corresponding \( y \)-value and each \( y \)-value has exactly one corresponding \( x \)-value. Only one-to-one functions have inverses that are functions. Such functions are said to be **invertible**. If we know the graph of a function, we can use the **Horizontal Line Test** to determine if the function is one-to-one.

**Horizontal Line Test.** If any horizontal line passes through a graph of a function more than once, then the function is **not** one-to-one.

If a function is not one-to-one, we sometimes choose a one-to-one branch of the function so that an inverse function is possible. For example, \( y = x^2 + 3 \) is not a one-to-one function. However, if we restrict the domain to \((0, \infty)\) by saying \( y = x^2 + 3 \), where \( x \geq 0 \), then we have a one-to-one branch and this function does have an inverse function.

The fact that the \( x,y \) pairs in a function and in its inverse are interchanged suggests a way of finding the formula for the inverse of a function whose formula is known. Simply interchange the \( x \)'s and \( y \)'s and solve for the new \( y \). For example, suppose \( y = f(x) = \sqrt[3]{\frac{2x - 5}{4}} \). Then to determine the formula for the inverse, we first interchange the \( x \)'s and \( y \)'s results in \( x = \sqrt[3]{\frac{2y - 5}{4}} \). Then to solve for the new

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y, we cube both sides to get \( x^3 = \frac{2y - 5}{4} \), then multiply both sides by 4 to get \( 4x^3 = 2y - 5 \), then add 5 to both sides to get \( 4x^3 + 5 = 2y \), and finally divide both sides by 2 to get \( y = \frac{4x^3 + 5}{2} \). This result is the formula for the inverse function, so \( f^{-1}(x) = \frac{4x^3 + 5}{2} \). You can verify algebraically, that for \( f(x) = \sqrt[3]{\frac{2x - 5}{4}} \) and \( f^{-1}(x) = \frac{4x^3 + 5}{2} \), it is indeed true that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \), as required for inverse functions.

The fact that the \( x,y \) pairs in a function and in its inverse are interchanged also determines how the graphs of the function and its inverse will be related. The graphs of \( f(x) \) and \( f^{-1}(x) \) are symmetric to each other about the diagonal line \( y = x \). In other words, if both \( f(x) \) and \( f^{-1}(x) \) are graphed on the same coordinate system and the graph is then folded along the diagonal \( y = x \), the two graphs will lie exactly on top of each other.