Polynomials

A function that can be put in the form \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), where \( n \) is a nonnegative integer and \( a_n \neq 0 \), is called an \( n \)-th degree polynomial. \( a_n x^n \) is called the leading term of the polynomial, \( a_n \) is called the leading coefficient, and \( a_0 \) is called the constant term. Writing the terms of a polynomial in descending order of the powers is standard form.

The graph of a polynomial is a smooth, connected curve. There will always be exactly one y-intercept and an \( n \)-th degree polynomial will have, at most, \( n \) x-intercepts and, will have at most, \( n-1 \) peaks and valleys (formally called "relative maxima" and "relative minima," respectively. The intercepts and "relative extrema" (peaks and valleys) of the graph of a polynomial are often referred to as the short-term behavior of the polynomial.

If \( p(x) \) is a polynomial, the solutions to the equation \( p(x) = 0 \) are called the zeros of the polynomial. Sometimes the zeros of a polynomial can be determined by factoring or by using the Quadratic Formula. Often the zeros must be approximated. The zeros of a polynomial are its x-intercepts. Because the graph of a polynomial is connected, if the polynomial is positive at one value of \( x \) and negative at another value of \( x \), then there must be a zero of the polynomial between those two values of \( x \).

The Factor Theorem: If \((x - k)\) is a factor of a polynomial, then \( x = k \) is a zero of the polynomial. Conversely, if \( x = k \) is a zero of a polynomial, then \((x - k)\) is a factor of the polynomial.

The number of times a factor occurs in a polynomial is called the multiplicity of the factor. The corresponding zero is said to have the same multiplicity. For example, if the factor \((x - 3)\) occurs to the fifth power in a polynomial, then \((x - 3)\) is said to be a factor of multiplicity 5 and the corresponding zero, \( x=3 \), is said to have multiplicity 5. A factor or zero with multiplicity two is sometimes said to be a double factor or a double zero. Similarly, a factor or zero with multiplicity three is sometimes said to be a triple factor or a triple zero.
If a factor of a polynomial occurs to an odd power, then the graph of the polynomial actually goes across the $x$-axis at the corresponding $x$-intercept. An $x$-intercept of this type is sometimes called an **odd $x$-intercept**. If a factor of a polynomial occurs to an even power, then the graph of the polynomial "bounces" against the $x$-axis at the corresponding $x$-intercept, but not does not go across the $x$-axis there. An $x$-intercept of this type is sometimes called an **even $x$-intercept**.

As a polynomial is graphed farther and farther to the left or right, the graph eventually turns up and continues up from then on or turns down and continues down from then on. This fact is referred to as the **long-term behavior** or **end behavior** of the polynomial. Some texts describe the "tails" of the graph of a polynomial in terms of the **ultimate direction** of the graph as $x \to \infty$ or $x \to -\infty$. Because as $x \to \infty$ or $x \to -\infty$, the leading term becomes dominant (gets far bigger in absolute value than any of the other terms), it is the leading term that determines the long-term behavior of a polynomial.

**Long Term Behavior of an Even Degree Polynomial**: If the degree of a polynomial is even and the leading coefficient is positive, both tails go up. If the degree of a polynomial is even and the leading coefficient is negative, both tails go down.

**Long Term Behavior of an Odd Degree Polynomial**: If the degree of a polynomial is odd and the leading coefficient is positive, the tail on the left goes down and the tail on the right goes up. If the degree of a polynomial is odd and the leading coefficient is negative, the tail on the left goes up and the tail on the right goes down.