ASYMPTOTES OF RATIONAL FUNCTIONS

\[ y = f(x) = \frac{N(x)}{D(x)} \]

where N(x) and D(x) are polynomials

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HORIZONTAL ASYMPTOTES, \( y = b \)

A horizontal asymptote is a horizontal line that is not part of a graph of a function but guides it for x-values “far” to the right and/or “far” to the left. The graph may cross it but eventually, for large enough or small enough values of x (approaching \( \pm \infty \)), the graph would get closer and closer to the asymptote without touching it. A horizontal asymptote is a special case of a slant asymptote.

A “recipe” for finding a horizontal asymptote of a rational function:

Let

\[ \text{deg } N(x) = \text{ the degree of a numerator and } \text{deg } D(x) = \text{ the degree of a denominator.} \]

<table>
<thead>
<tr>
<th>( \text{deg } N(x) = \text{deg } D(x) )</th>
<th>( \text{deg } N(x) &lt; \text{deg } D(x) )</th>
<th>( \text{deg } N(x) &gt; \text{deg } D(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{\text{leading coefficient of } N(x)}{\text{leading coefficient of } D(x)} )</td>
<td>( y = 0 ) which is the x-axis</td>
<td>There is no horizontal asymptote</td>
</tr>
</tbody>
</table>

Another way of finding a horizontal asymptote of a rational function:

Divide N(x) by D(x). If the quotient is constant, then \( y = \) this constant is the equation of a horizontal asymptote.

**Examples**

**Ex. 1**

\[ y = \frac{-2x^3 - 3x + 5}{x^3 + 1} = -2 + \frac{-3x + 7}{x^3 + 1} \]

HA: \( y = -2 \)
because \( \frac{-3x + 7}{x^3 + 1} \) approaches 0 as x increases.

**Ex. 2**

\[ y = \frac{2x + 1}{x} = 2 + \frac{1}{x} \]

HA: \( y = 2 \)
because \( \frac{1}{x} \) approaches 0 as x increases.

**Ex. 3**

\[ y = \frac{3x^2}{x + 1} = (3x - 3) + \frac{3}{x + 1} \]

approaches \( \infty \) as x increases (\( y = 3x - 3 \) is a slant asymptote.)

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By Joanna Gutt-Lehr, Pinnacle Learning Lab, last updated 1/2010
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SLANT (OBLIQUE) ASYMPTOTE, \( y = mx + b, m \neq 0 \)

A slant asymptote, just like a horizontal asymptote, guides the graph of a function only when \( x \) is close to \( \pm \infty \) but it is a slanted line, i.e. neither vertical nor horizontal. A rational function has a slant asymptote if the degree of a numerator polynomial is 1 more than the degree of the denominator polynomial.

A “recipe” for finding a slant asymptote of a rational function:

Divide the numerator \( N(x) \) by the denominator \( D(x) \). Use long division of polynomials or, in case of \( D(x) \) being of the form: \((x-c)\), you can use synthetic division.

The equation of the asymptote is \( y = mx + b \) which is the quotient of the polynomial division (ignore remainder)

Examples

\[
f(x) = \frac{6x^3 - 1}{-2x^2 + 18}
\]

\( \text{deg } N(x) = 3, \quad \text{deg } D(x) = 2. \)

Perform long division \( D(x) \big| N(x) : \)

\[
\begin{align*}
-3x & \\
-6x^3 + 54x & \\
54x - 1 & \quad \leftarrow \text{this is the remainder}
\end{align*}
\]

\[ f(x) = -3x + \frac{54x - 1}{-2x^2 + 18} \]

The slant asymptote’s equation is:

\[ y = -3x \]

\[
f(x) = \frac{2x^2 + x - 5}{x + 1}
\]

\( \text{deg } N(x) = 2, \quad \text{deg } D(x) = 1. \)

Perform synthetic division:

\[
\begin{array}{c|cc}
 \text{Zero of the denominator} & 2 & 1 & -5 \\
 \text{ } & -2 & 1 \\
 \hline
 2 & -1 & 4 \quad \leftarrow \text{this is the remainder}
\end{array}
\]

\[ f(x) = 2x - 1 + \frac{-4}{x + 1} \]

The slant asymptote’s equation is:

\[ y = 2x - 1 \]

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