Definitions and Laws of Exponents

Definition of Positive Integer Exponents
For any number \( b \) and any positive integer \( n \), \( b^n \) stands for the product of \( n \) factors of \( b \). In general, \( b^n \) is called “the \( n \)-th power of \( b \)” or “\( b \) to the \( n \)-th power.” For example, \( 2^5 \) stands for the product of 5 factors of 2. So \( 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32 \). \( 2^5 \) is called “the fifth power of 2” or “2 to the fifth power.” The second power of a number is usually called its square so \( 7^2 \) is usually called “7 squared.” The third power of a number is usually called its cube so \( 4^3 \) is usually called “4 cubed.”

Note: An exponent applies only to the number it is immediately to the right of unless there are parentheses; in that case the exponent applies to everything in the adjacent parentheses.

Examples: \(-5^2 = -5 \cdot 5 = -25 \) whereas \((-5)^2 = (-5)(-5) = 25\).

Laws of Exponents

Product Rule. \( b^m \cdot b^n = b^{m+n} \). In other words, to multiply two powers with the same base, add the exponents and write that sum as the exponent on the base. For example, \( x^2 \cdot x^3 = x^{2+3} = x^5 \).

Quotient Rule. \( b^m + b^n = b^{m-n} \). In other words, to divide two powers with the same base, subtract the exponents and write that difference as the exponent on the base. For example, \( \frac{x^{10}}{x^7} = x^{10-7} = x^7 \).

Power of a Power Rule. \( (b^n)^m = b^{mn} \). In other words, to determine a power of a power, multiply the exponents and write that product as the exponent on the base. For example, \( (x^7)^6 = x^{42} \).

Power of a Product Rule. \( (ab)^n = a^n b^n \). In other words, to determine a power of a product, apply the exponent to each factor in the product. For example, \( (5x)^2 = 5^2 x^2 = 25x^2 \) and \( (p^2 s^3)^4 = (p^8 s^{12}) = p^{15} s^{20} \).

Power of a Quotient Rule. \( \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \). In other words, to determine a power of a quotient, apply the exponent to both the numerator and denominator. So \( \left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3} = \frac{8}{125} \) and \( \frac{v^4}{w^3} = \frac{(v^4)^9}{w^9} = v^{36}w^{-9} \).

Definition of Negative Exponents
A negative power of a number is equal to the reciprocal of the positive power of the number; that is, \( b^{-n} = \frac{1}{b^n} \). For example, \( 2^{-3} = \frac{1}{2^3} = \frac{1}{8} \) and \( x^{-5} = \frac{1}{x^5} \).

As a result of this definition, it also follows that a negative power of a fraction is equal to the positive power of the reciprocal of the fraction; in other words, \( \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \). For example,

\[
\left(\frac{4}{5}\right)^{-3} = \left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64} \quad \text{and} \quad \left(\frac{x^3}{y^2}\right)^{-6} = \left(\frac{y^2}{x^3}\right)^6 = \frac{y^{12}}{x^{18}}.
\]

Note: The Laws of Exponents given above are true for all kinds of exponents, including negative ones. However, answers are usually given using only positive exponents. For example,

\[
x^{-2} \cdot x^{-5} = x^{-2+(-5)} = x^{-7} = \frac{1}{x^7}, \quad \frac{x^{-3}}{x^5} = x^{-3-6} = x^{-9} = \frac{1}{x^9}, \quad \text{and} \quad x^{-3} y^{-2} = (x^{-4})^{-2} (y^3)^{-2} = x^{(-4)(-2)} y^{3(-2)} = x^8 y^{-6} = \frac{x^8}{y^6}.
\]

(Thomason, Spring 2015)