# Gauss-Jordan Elimination for Solving a System of \( n \) Linear Equations with \( n \) Variables

To solve a system of \( n \) linear equations with \( n \) variables using Gauss-Jordan Elimination, first write the augmented coefficient matrix.

The general idea is as follows: Work across the columns from left to right using Elementary Row Operations to first get a 1 in the diagonal position and then to get 0’s in the rest of that column. When the left-most \( 3 \times 3 \) sub-matrix has 1’s on its upper-left-to-lower-right diagonal and 0’s elsewhere, the solutions are in the fourth column of the main \( 3 \times 4 \) matrix.

For the \( 3 \times 3 \) the system
\[
\begin{align*}
2x - 2y + 4z &= 10 \\
3x - 6y + 6z &= 18 \\
-2x + 5y - 3z &= -11
\end{align*}
\]
the augmented coefficient matrix is
\[
\begin{bmatrix}
2 & -2 & 4 & 10 \\
3 & -6 & 6 & 18 \\
-2 & 5 & -3 & -11
\end{bmatrix}
\]

Starting with the 1st column, if the diagonal position is 0, swap the 1st row with a lower row that doesn’t have a 0 in the first column. When the diagonal is not 0, multiply the 1st row by the reciprocal of the 1st row, 1st column entry to get a 1 in the diagonal position.

<table>
<thead>
<tr>
<th>[ \frac{1}{2} R_1 \rightarrow R_1 ]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & -1 & 2 & 5 \\
3 & -6 & 6 & 18 \\
-2 & 5 & -3 & -11
\end{bmatrix}
\]

Use multiples of the 1st row to zero out the other entries in the 1st column.

<table>
<thead>
<tr>
<th>[-3R_1 + R_2 \rightarrow R_2 ]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & -1 & 2 & 5 \\
0 & -3 & 0 & 3 \\
0 & 3 & 1 & -1
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>[2R_1 + R_3 \rightarrow R_3]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & -1 & 2 & 5 \\
0 & 3 & 1 & -1
\end{bmatrix}
\]

Next we work on the 2nd column. We want to have a 1 in the diagonal position so multiply the 2nd row by \(-\frac{1}{3}\).

<table>
<thead>
<tr>
<th>[-\frac{1}{3} R_2 \rightarrow R_2]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & -1 & 2 & 5 \\
0 & 1 & 0 & -1 \\
0 & 3 & 1 & -1
\end{bmatrix}
\]

Then we use multiples of the 2nd row to zero out the other entries in the 2nd column.

<table>
<thead>
<tr>
<th>[R_2 + R_1 \rightarrow R_1]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>[-3R_2 + R_3 \rightarrow R_3]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

Finally we work on the 3rd column. We want a 1 on the diagonal and, as fate would have it, there’s already a 1 in that position. So all we have left to do is use multiples of the 3rd row to zero out the rest of the 3rd column. Then the matrix is in Reduced Row Echelon Form and we see that the solution is \((0,-1,2)\).

<table>
<thead>
<tr>
<th>[-2R_3 + R_1 \rightarrow R_1]</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]
What can possibly “go wrong”?

What, if when you get to the last column, it is impossible to get a 1 on the diagonal because there is a 0 in that position. Therefore it is impossible to zero-out the rest of the last column using multiples of the last row.

There are two possibilities we’ll consider.

**Situation 1 – All of the entries in the bottom row are 0’s.**

Example:

\[
\begin{bmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The bottom row indicates that \(0x + 0y + 0z = 0\), which is true for any values of the variables. In other words, there are an infinite number of solutions. (The system is dependent.)

**Situation 2 – All of entries in the bottom row are 0’s except for the last entry.**

Example:

\[
\begin{bmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 9
\end{bmatrix}
\]

The bottom row indicates that \(0x + 0y + 0z = 9\), which is not true for any values of the variables. In other words, there are no solutions. (The system is inconsistent.)