# Gauss-Jordan Elimination

for Solving a System of $n$ Linear Equations with $n$ Variables

To solve a system of $n$ linear equations with $n$ variables using *Gauss-Jordan Elimination*, first write the *augmented coefficient matrix*.

The general idea is as follows: Work across the columns from left to right using Elementary Row Operations to first get a 1 in the diagonal position and then to get 0’s in the rest of the column. When the left-most $3 \times 3$ sub-matrix has 1’s on its diagonal and 0’s elsewhere, the solutions are in the fourth column of the main matrix.

### For the $3 \times 3$ the system

$$
\begin{align*}
2x - 2y + 4z &= 10 \\
3x - 6y + 6z &= 18 \\
-2x + 5y - 3z &= -11
\end{align*}
$$

the augmented coefficient matrix is

$$
\begin{bmatrix}
2 & -2 & 4 & 10 \\
3 & -6 & 6 & 18 \\
-2 & 5 & -3 & -11
\end{bmatrix}
$$

### Starting with the first column, if the diagonal position is 0, swap the first row with a lower row that doesn’t have a 0 in the first column. When the diagonal is not 0, multiply the first row by the reciprocal of the first row, first column entry to get a 1 in the diagonal position.

- $\frac{1}{2} R_1 \rightarrow R_1$

### Next we work on the second column. We want to have a 1 in the diagonal position so multiply the second row by $-\frac{1}{3}$.

- $-3R_1 + R_2 \rightarrow R_2$
- $2R_1 + R_3 \rightarrow R_3$

### Finally we work on the third column. We want a 1 on the diagonal and, as fate would have it, there’s already a 1 in that position. So all we have left to do is use multiples of row 3 to zero out the rest of the third column. Then the matrix is in *Reduced Row Echelon Form* and we see that the solution is $(0,-1,2)$.

- $R_2 + R_1 \rightarrow R_1$
- $-3R_2 + R_3 \rightarrow R_3$

- $-2R_3 + R_1 \rightarrow R_1$

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What can possibly “go wrong”?

What, if when you get to the last column, it is impossible to get a 1 on the diagonal because there is a 0 in that position. Therefore it is impossible to zero-out the rest of the last column using multiples of the last row.

There are two possibilities we’ll consider.

**Situation 1 – All of the entries in the bottom row are 0’s.**

Example:

\[
\begin{bmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The bottom row indicates that \(0x + 0y + 0z = 0\), which is true for any values of the variables. In other words, *there are an infinite number of solutions*. (The system is dependent.)

**Situation 2 – All of entries in the bottom row are 0’s except for the last entry.**

Example:

\[
\begin{bmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 3 & -4 \\
0 & 0 & 0 & 9
\end{bmatrix}
\]

The bottom row indicates that \(0x + 0y + 0z = 9\), which is not true for any values of the variables. In other words, *there are no solutions*. (The system is inconsistent.)