

Review of Linear Relationships

Mary Parker, January 25, 2010

Objectives:

1. Given an appropriate verbal description of a linear relationship, make a table of values, a graph, and write the formula for the relationship. Use whatever letters to stand for the variables that are given in the problem. If none are given, you may use x and y .
2. Given points on a line, find the formula for the line. It is easiest to use x and y for the variables when you do this, but then be able to translate the resulting formula into other variables if requested.
3. Interpret the slope and y -intercept of a line. Use the names and units of the variables in the problem (such as time and cost) if that information is available.
4. From the graph of a line, estimate the formula for the line.

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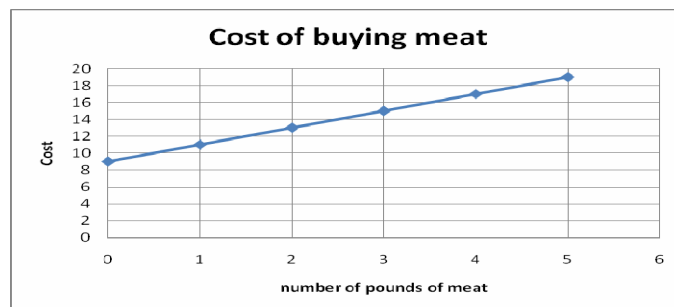
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Section 1: Stories, tables, and graphs.

Example 1. (Buying meat) Jean needs to buy some meat for her housing co-op. She can buy it at a food co-op in bulk. She knows that a trip to the food co-op costs about \$9 (in terms of her time and money to run the car.) The meat she is to buy costs \$2 per pound. For the purpose of this example, all she will buy on this trip to the food co-op is meat. The following table and graph summarize the cost for several different amounts of meat, where n is the number of pounds of meat she buys and C is the cost, including the cost of going to the food co-op.

n	C
0	\$9
1	$\$9 + \$2 = \$11$
2	$\$9 + 2(\$2) = \$13$
3	$\$9 + 3(\$2) = \$15$
4	$\$9 + 4(\$2) = \$17$
5	$\$9 + 5(\$2) = \$19$



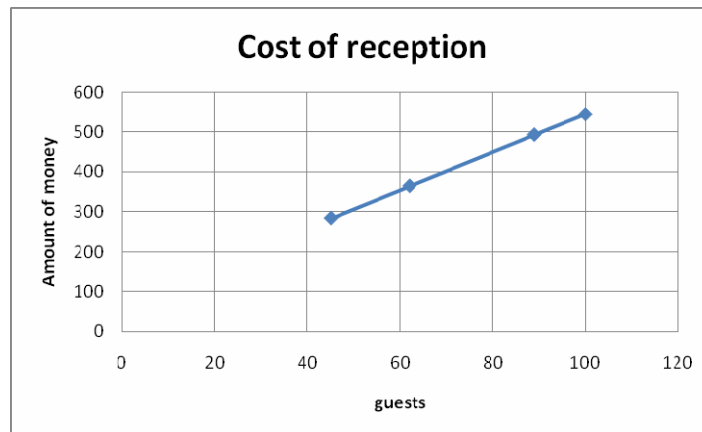
Example 2: (Buying eggs) Salima needs to buy some eggs for her housing co-op. She can buy it at a food co-op in bulk. She knows that a trip to the food co-op costs about \$7 (in terms of her time and money to run the car.) The eggs she is to buy costs \$3 per dozen. For the purpose of this example, all she will buy on this trip to the food co-op is eggs. The following table and graph summarize the cost for several different amounts of eggs, where x is the number of dozen eggs she buys and y is the cost, including the cost of going to the food co-op.

x	y
0	\$7
1	$\$7 + \$3 = \$10$
2	$\$7 + 2(\$3) = \$13$
3	$\$7 + 3(\$3) = \$16$
4	$\$7 + 4(\$3) = \$19$
5	$\$7 + 5(\$3) = \$22$



Example 3: Jeremy is planning a reception at his art gallery and needs to hire a caterer. He knows that the caterer charges a certain amount and then an additional amount per guest. He has five friends who have recently used this caterer and has information about how much they paid. This is recorded in the table below, where g is the number of guests and A is the total amount of money paid to the caterer.

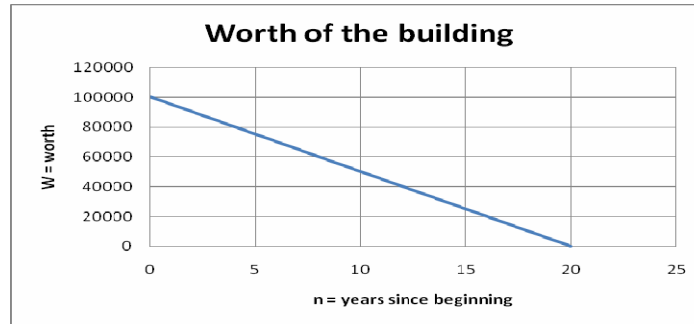
g	A
45	\$283.75
62	\$364.50
89	\$492.75
100	\$545



Practice problem 1. Jeremy has another choice for his reception. This same caterer will prepare a gourmet menu. If there are 40 guests, the cost of a gourmet reception is \$399; if there are 50 guests, the cost of a gourmet reception is \$476.50, and if there are 80 guests, the cost of a gourmet reception is \$709. Assign appropriate variables, define your variables, make a table with appropriate labels, and make a graph with appropriate scales on both the horizontal and vertical axis.

Example 4. For accounting purposes, owners of buildings often depreciate their value over some period of time. Usually this is linear depreciation. So, if we use linear depreciation to depreciate a building that is worth \$100,000 over a period of 20 years, then we have it worth \$100,000 at the beginning, which is 0 years, and worth \$0 at the end, which is at 20 years. Let t stand for the number of years since the beginning and W be the worth of the building according to the depreciation model.

t	W
0	\$100,000
20	\$0



Section 2. Find Formulas

This section includes two methods of finding the formula for a linear relationship (finding the formula from the pattern and finding the formula from two points) and it also includes interpreting the coefficients in the formula.

2.1 Write a formula from the pattern

Example 5. (continuation of Example 1, about buying meat.) Suppose Jean needs to buy 20 pounds of meat. Can we find a formula to use that we can simply plug in numbers to find the cost? YES!

Notice that, for each additional pound of meat, the cost is an additional \$2. So we can look at a chain of computations and see the pattern, and extend that pattern as far as we want, and even extend it by writing a general formula. (Here we will drop the dollar signs since they just make the formula look more complicated and we know that the costs are all in dollars.)

If $n = 0$, then $C = 9$.

If $n = 1$, then $C = 9 + 2$.

If $n = 2$, then $C = 9 + 2 + 2 = 9 + 2(2)$.

If $n = 3$, then $C = 9 + 3(2)$.

If $n = 4$, then $C = 9 + 4(2)$.

If $n = 20$, then $C = 9 + 20(2)$.

etc.

For n , then $C = 9 + n(2) = 9 + 2n$

Practice problem 2: For Example 2, about buying eggs, find a formula for y in terms of x . Use all the steps we did in Example 4 above.

2.2. Identify and interpret the slope and y-intercept

In algebra classes, we learned that the equation of a line is $y = mx + b$ where x and y are the variables, and m and b are numbers. We also learned the names of m and b : m is the slope and b is the y-intercept.

Example 6 (continuation of Example 5.) For the cost of meat problem, is this a linear relationship, and, if so, what are the slope and intercept?

Answer: Yes, this is a linear relationship. We can see that from the graph because the points make a straight line. We could see that from the table even before we made the graph, because, for x 's that increase by the same amount each time (here they increase by 1) then the y 's increase by the same amount each time (here they increase by 2.) That means the relationship is linear.

The formula we found is $C = 9 + 2n$. This is the same relationship as $y = 9 + 2x$. (We see that C is in the place of y and n is in the place of x .) We learned in arithmetic class that addition is commutative, so $C = 9 + 2n$ is equivalent to $C = 2n + 9$. Since the slope is the number multiplied by the variable in the place of x , we see that the slope is 2 and the y-intercept is 9. To summarize, the formula $C = 9 + 2n$ is a linear formula with the variables C and n , with slope 2 and y-intercept 9.

Interpret the slope and y-intercept.

y-intercept. Look at the graph in Example 2, where Salima goes to the food co-op to buy eggs. Recall that the y-intercept of a graph is the y -value where the graph crosses the y -axis. For that example, the graph crosses at 7. And since the y -axis is at the place where $x = 0$, notice that we could get that also by looking at the table or the formula and finding the y -value for $x = 0$. The y-intercept of a graph is the y -value when $x = 0$. In the words of the story for Example 2, the y-intercept is the cost of going to the store if Salima didn't buy any eggs. (Maybe she went to the store and they were out of eggs.)

Slope: The slope of a graph is the amount that y increases when x increases by 1 unit. From algebra class, we recall that we can find the slope of a graph by $m = \frac{\text{rise}}{\text{run}}$. The rise is the change in the y -value and the run is the change in the x -value. In the words of the story for Example 2, for every additional carton of eggs Salima buys, the cost of her trip increases by \$3.

Comment about the slope: The relationship is linear BECAUSE the amount that y increases when x increases by 1 stays the same over the entire domain of values for x that we are interested

in. We will see some relationships in this course that are not linear. Then their formulas don't have what we call a slope coefficient.

Practice problem 3: In example 1, write an interpretation of the y -intercept in the words of the story and then write an interpretation of the slope in the words of the story.

2.3. Find the slope of the line from two points on the line.

To find the formula of the line, first find the slope and then find the y -intercept, and then put those values into the equation $y = mx + b$ where x and y are the variables, and m is the slope and b is the y -intercept.

In your algebra classes, you learned to find the slope of a line by taking two points on it and using the slope formula.

If the points (x_1, y_1) and (x_2, y_2) are on a line, the slope of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

When you use this formula, it doesn't matter which of the two points you call the first and which you call the second. It is important that you put the values for y in the numerator and the values for x in the denominator and that you keep the order the same in both the numerator and denominator.

When we are using these algebraic formulas, which have values for y and x , usually we write the equation in terms of y and x even if the original variables in the problem were different. Then after we have found the formula, we can rewrite it in terms of the original variables in the problem.

Example 7. (continuation of Example 3, about the cost of a reception) In this problem, we do not have x -values that increase by 1 each time, so we cannot write the formula by just working out the pattern as we could for Examples 1 and 2. We can pick any two points and use them in either order to find the slope. We choose the first two points. So we use $(x_1, y_1) = (45, 283.75)$

and $(x_2, y_2) = (62, 364.5)$. Then $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{364.50 - 283.75}{62 - 45} = \frac{80.75}{17} = 4.75$. We found that

the slope of this line is 4.75. In the words of the story, this means that the cost of the reception increases by \$4.75 for each additional guest.

Practice problem 4: For the same data about the cost of the reception in Example 3, pick a different pair of points and compute the slope. Is it the same as we computed in Example 7? Try it with a different pair. Keep trying these until you are convinced that it doesn't make any difference which two points you use, and also that the order of the points doesn't make any difference in the resulting slope.

Practice problem 5: For Practice problem 1 (gourmet reception) find and interpret the slope.

2.4 After finding the slope, use that and one point to find the y-intercept.

Example 8: (continuation of Examples 3 and 7) In the previous example, we used two points to find the slope. Now we will **use that slope and one of the points on the line** to find the value for the intercept. We plug in the slope for m in the equation and the two coordinates of the points for x and y . That leaves us with an equation where only one value is unknown, and that is b . We can solve it for b .

We will use $(x, y) = (62, 364.5)$ and the fact that $m = 4.75$.

$$\begin{aligned} y &= mx + b \\ 364.5 &= 4.75 \cdot 62 + b \\ 364.5 &= 294.5 + b \\ 364.5 - 294.5 &= 294.5 + b - 294.5 \\ 70 &= b \end{aligned}$$

2.5. Use the slope and intercept to write the formula for the line.

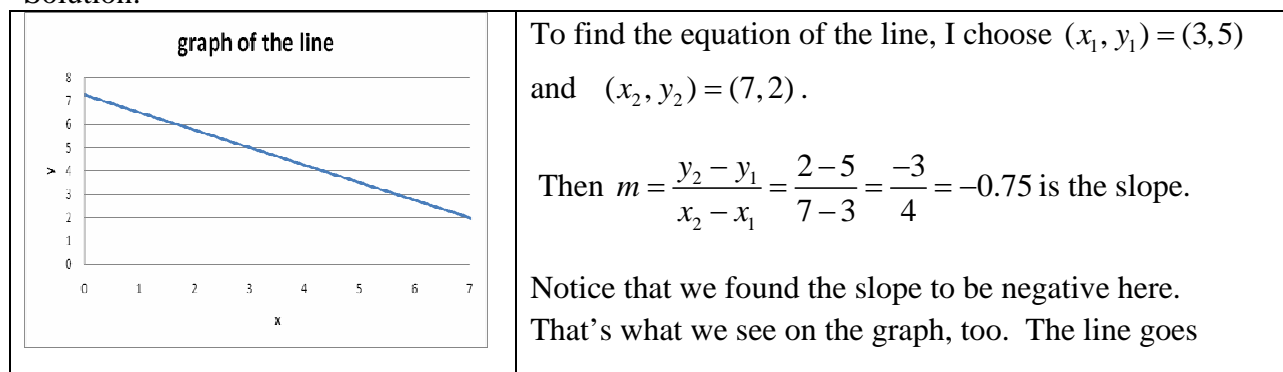
Now that we have $m = 4.75$ and $b = 70$ we can write the equation: $y = 4.75x + 70$.

Since the original problem has the variables g for the number of guests and A for the amount of money it will cost, then we can write this formula for this relationship as $A = 4.75g + 70$

Practice problem 6: For Practice Problem 1 (gourmet reception) find the y-intercept, use the slope from the previous practice problem to write the formula for the relationship in terms of y and x and then write the formula for the relationship in terms of the variables in the original story.

Example 9: Graph the two points $(7,2)$ and $(3,5)$. Draw the line through those two points. Then Find the equation of the line through the points and interpret the slope and intercept.

Solution:



	down, which says that, as x increases, y decreases.
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Comment: The line goes down, which says that, as x increases, y decreases. Another way of saying that is that the “rise” here is really a “fall”. But, in math, we still call it a “rise” and make it a negative number, to indicate that it goes in the opposite direction.

Use the point $(x, y) = (3, 5)$ and this slope to find the y -intercept.

$$\begin{aligned}
 y &= mx + b \\
 5 &= -0.75 \cdot 3 + b \\
 5 &= -2.25 + b && \text{The equation of the line is } y = 7.25 - 0.75x. \\
 5 + 2.25 &= -2.25 + b + 2.25 \\
 7.25 &= b
 \end{aligned}$$

Interpretation of the slope: If x increases by 1, then y decreases by 0.75.

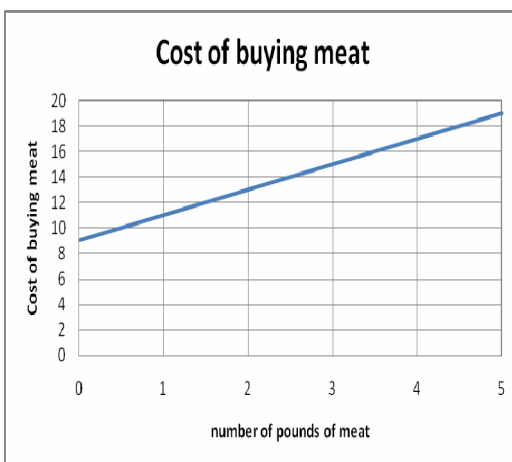
(Comment: This is what we saw on the graph – as x increases, y decreases. In this interpretation, we even say how much y decreases by.)

Interpretation of the y -intercept: If x is zero, then y is 7.25.

Practice problem 7: For Example 5 (the depreciation problem,) find the formula for the line and then interpret the slope and intercept in terms of years since the beginning and worth of the building.

Section 3. Estimate the slope and intercept of a line from the graph

When we have a graph of a linear formula, we can find the y -intercept by just looking at where the graph crosses the y -axis. Below is the graph from Example 1 and the formula for that relationship that we found in Example 4. See that the graph crosses the y -axis at the point $(0, 9)$, so that the y -intercept is 9. That’s what we see in the formula as well – that the y -intercept is 9.



Same relationship, from Example 5:

$$C = 9 + 2n$$

Here we see that the y -intercept of the graph is 9. That fits with the equation above.

When we look at the graph to estimate the slope, we need to estimate the “rise” and the “run” and divide those. See the discussion below to see that we estimate that to be 2. That also fits with the equation above.

When we look at the graph to estimate the slope, we need to estimate the “rise” and the “run” and divide those. We can estimate the position of two points and use those to find the rise and the run. I choose the points (1, 11) and (4, 17). Just from the graph, I can’t tell those y-values for sure – the first point might really be (1, 11.2) or something like that. But I choose to read the graph as saying that point is (1,11). So, looking at the graph, the run is from 1 to 4, so that is $\text{run} = 4 - 1 = 3$. The rise is from 11 to 17, so the rise is $17 - 11 = 6$. Thus, the slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = 2. \text{ That is consistent with what we see in the equation.}$$

How accurate? Suppose you have worked a problem like this and then looked up the answer in the answer key. And your answer is not exactly the same. How can you check your work?

Answer: Your **y-intercept** should be pretty close to the one in the answer key. How close?

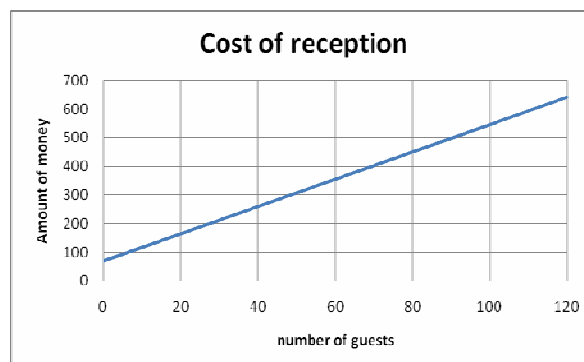
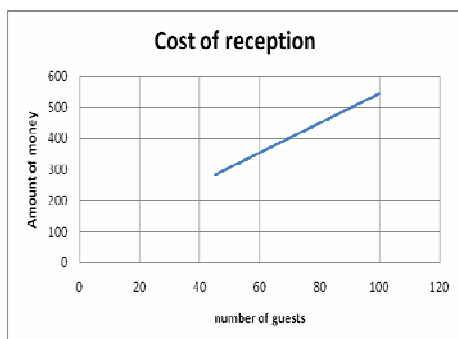
Well, look at the graph. If you put a dot on the y-axis for your y-intercept and a dot on the y-axis for the answer key’s y-intercept, they should look pretty close together.

Your **slope** must have the same sign (positive or negative) as the slope in the answer key. How far off is OK depends on the size of the numbers. For a hand-drawn graph, you should be able to estimate the slope within 20% of the correct value. So, if the correct value is $m = 4$, then your estimate should be within $0.20 \cdot 4 = 0.8$ of that correct value. So any number between 3.2 and 4.8 is a reasonable estimate.

Practice problem 8: From the graph in Example 2 about buying eggs, estimate the slope and intercept and write the formula for the line. Then discuss whether your estimate is accurate enough by comparing it to the formula found for this relationship in Example

Sometimes the graph you are given doesn’t extend to the y-axis. So you will need to use a straight-edge (like a ruler) and extend it by hand to estimate the intercept. “Estimate” is a very important word here. No one expects you to have really precise answers when you are estimating. Just do the best you can.

Here’s the graph from Example 3 and an extension of it that my hand-drawn graph would show.



From the extended graph, I estimate that the y -intercept is 80 and that two points on the graph are (40,260) and (100,550). So I estimate that $m = \frac{\text{rise}}{\text{run}} = \frac{290}{60} = 4.83$. Thus my estimated equation of the line is $y = 4.83x + 80$, and, in terms of the variables in the given story $A = 4.83g + 80$.

Let's evaluate how good our estimate is.

First, we must find the correct answer. In Examples 7 and 8, we used the exact points given to find the formula $A = 4.75g + 70$.

First, look at the y -intercept. When I put a dot on the y -axis of the graph at the correct value of 70 and another at my estimate of 80, they look very close. So this is a good estimate. Now look at the slope. The correct value is positive and we found a positive slope. So that's good. We compute 20% of the correct value as $0.20 \times 4.75 = 0.95$. So any value between 4.75 ± 0.95 would be an acceptable estimate for a hand-drawn graph on graph paper. Our estimate here is clearly within that interval. This is an acceptable estimate for the formula of the line.

Practice problem 9: For the gourmet reception described in Practice Problem 1 (just after Example 3) use the graph you obtained in Practice Problem 1 and estimate the intercept and slope just by looking at the graph (instead of looking at the exact points you plotted.) Use those to write your estimate of the formula for the line. Then compare your estimate with the correct answer for the formula you found in Practice Problem 5. Explain whether your estimate meets the criteria for a good estimate.

Section 3. What more will we need to do with linear relationships?

Application to real data: What if we have some real data, and so the graph isn't exactly linear, but it is pretty close to linear? Can we work with that?

Short answer: Yes. We can sketch a line on the graph that looks pretty close to the points, compute the equation of that line, and use that to summarize the relationship. We'll use this idea in all three of the Maricopa modules.

Function notation: What if we find it tedious to always have to make a table to show the pairs of numbers clearly? Can we find a notation that allows us to show this in a briefer way?

Short answer: Yes. In function notation for example 1, we would write $C(3) = 15$ to indicate that the cost for Jean to purchase three pounds of meat is \$15. I'll provide a handout with more explanation. We'll use this notation in the Exponential Growth module.

Finding the intersection point of two lines: Do you remember learning to solve a system of two equations in two variables? That solution point is the intersection point of the two lines. In

algebra class we learned to solve these both by elimination and by substitution. We will use this in the Systems module, and mostly we will use the method of substitution.

Section 4. Exercises.

Because this is review material, it is not clear how many problems you may need to work to practice these techniques. Thus, several sets are provided.

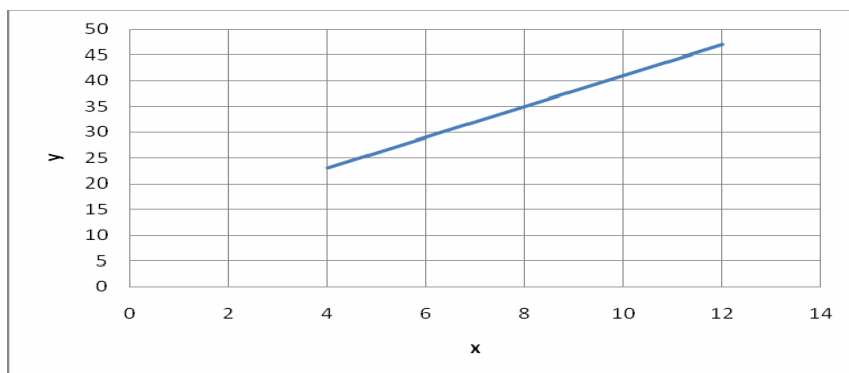
Exercise Set 1: Practice Problems 1-9 appear on pages 1- 9 of this handout, mixed in with the explanations. This is to strongly encourage you to read the explanations! Solutions are available at the end of this handout.

Exercise Set 2: (Quiz problems) These exercises below do not have answers provided except as provided by your instructor after you turn them in.

4.1. A bicycle manufacturer has daily fixed costs of \$1500 and it costs \$80 to manufacture one bicycle. Let n be the number of bicycles manufactured in one day and C be the manufacturer's cost for one day. Make a table to illustrate the costs of manufacturing 0, 1, 2, 3, 4, 5, or 6 bicycles in one day. Graph this relationship and use the method of finding a pattern to write the formula for this relationship. Interpret the slope and y-intercept of your formula.

4.2. Using graph paper, graph the line through $(7,13)$ and $(12,5)$. From your graph, estimate the y-intercept and tell whether the slope is positive or negative. Then find the equation of the line from these two points. Interpret the slope and y-intercept.

4.3. For the following linear graph, estimate the formula for the line. When your instructor gives you the accurate formula for this line, determine whether your estimate fit the conditions for being good enough.



The following sets of exercises are available on the course website beside this handout.

Optional Exercise Set 3: Additional exercises on finding the equation of the line from graphs. Answers are available.

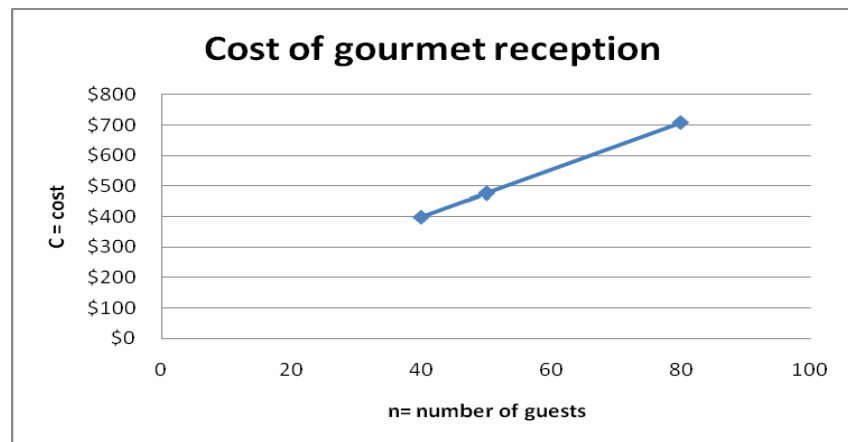
Optional Exercise Set 4: Additional exercises on finding the formula for a line algebraically, and interpreting the slope and intercept. These problems are examples in two of the review topics in the MATH 1333 materials. Since they are examples, they are completely worked out there.

Exercise Set 1. Solutions

Practice Problem 1.

Let n be the number of guests and C be the cost.

n	C
40	\$399
50	\$476.50
80	\$709



Practice Problem 2.

If $x = 0$, then $y = 7$.

If $x = 1$, then $y = 7 + 3$.

If $x = 2$, then $y = 7 + 3 + 3 = 7 + 2(3)$.

If $x = 3$, then $y = 7 + 3(3)$.

If $x = 4$, then $y = 7 + 4(3)$.

If $x = 20$, then $y = 7 + 20(3)$.

etc.

For x , then $y = 7 + x(3) = 7 + 3x$

Practice Problem 3.

y -intercept. If Jean goes to the store and buys nothing, the Cost is \$9.

Slope. For each additional pound of meat that Jean buys, the Cost of the shopping expedition increases by \$2.

Practice Problem 4.

Your work depends on which points you pick. Here are two examples.

First: $(x_1, y_1) = (45, 283.75)$ and $(x_2, y_2) = (100, 545)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{545 - 283.75}{100 - 45} = \frac{261.25}{55} = 4.75$$

Second: $(x_1, y_1) = (62, 364.5)$ and $(x_2, y_2) = (45, 283.75)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{283.75 - 364.50}{45 - 62} = \frac{-80.75}{-17} = 4.75.$$

Keep going as needed until it is clear to you that all pairs of points give the same slope.

Practice Problem 5:

Slope: $(x_1, y_1) = (40, 399)$ and $(x_2, y_2) = (80, 709)$. Then $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{709 - 399}{80 - 40} = \frac{310}{40} = 7.75$

Interpret: If we add one more guest, the cost of the gourmet reception increases by \$7.75.

Practice Problem 6:

We will use $(x, y) = (40, 399)$ and the fact that $m = 7.75$.

$y = mx + b$ The equation, in terms of x and y , is

$$399 = 7.75 \cdot 40 + b$$

$$399 = 310 + b$$

$$399 - 310 = 310 + b - 310$$

$$89 = b$$

$$y = 89 + 7.75x$$

In terms of the variables we chose for this problem in Practice Problem 1, n for the number of guests and C for the cost, we have

$$C = 89 + 7.75n$$

Practice Problem 7:

Slope: $(x_1, y_1) = (0, 100000)$ and $(x_2, y_2) = (20, 0)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 100000}{20 - 0} = \frac{-100000}{20} = -5000.$$

$$y = mx + b$$

We will use $(x, y) = (0, 100000)$ and the fact that $m = -5000$.

$$100000 = -5000 \cdot 0 + b$$

$$100000 = 0 + b$$

$$100000 = b$$

The equation, in terms of x and y , is $y = 100,000 - 5000x$

In terms of the variables in Example 3, t for the number of years since the beginning and W for the worth of the building, we have $W = 100,000 - 5000t$.

Practice Problem 8:

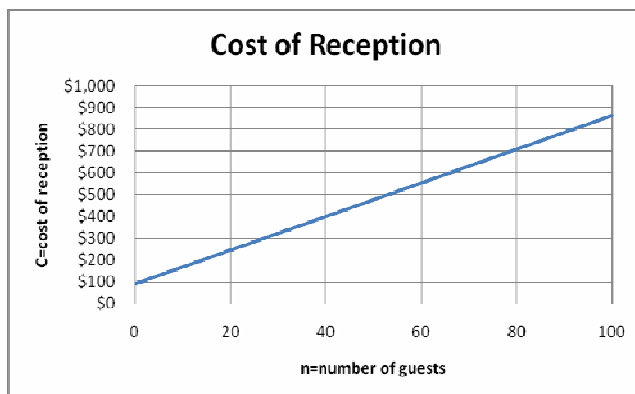
From Example 2, I looked at the graph and estimated that the y-intercept was 7.5, since it is about halfway between 5 and 10. I estimated two points to be (1,10) and (5,21). So my estimate of the rise is $21-10=11$ and my estimate of the run is $5-1=4$. So my estimate of the slope is $11/4 = 2.75$. So my estimate for the formula for the line is $y = 7.5 + 2.75x$.

Check: I recall that, in the discussion after Example 6, we found that the y-intercept for this relationship is 7 and the slope for this relationship is 3. When I put my answer for the y-intercept of 7.5 and the correct answer for the y-intercept of 7 on the y-axis of the graph, the points are very close. My estimate for the y-intercept is good.

Slope: The correct answer for the slope is positive and so is my estimate. That's good.

When I look at the correct answer for the slope and find 20% of it, I have $0.20 \times 3 = 0.6$, so the interval that the estimate should be in is $3 \pm 0.6 = 2.4$ to 3.6 . My answer of 2.75 is in this interval, so that's good.

Practice Problem 9: First I extend the graph to the y-axis.



y-intercept: I estimate it is 100.

Points: I estimate (20,250) and (60,550) are points on the graph so the rise is $550-250=300$ and the run is $60-20=40$, so the slope is $300/40 = 7.5$

My estimate: $C = 100 + 7.5n$

Check: The correct answer, from Practice Problem 6, is $C = 89 + 7.75n$.

y-intercept: When I put the correct answer of 89 and my estimate of 100 on the graph, they don't look very different. My estimate is close enough.

Slope: The correct answer is positive and so is my estimate, so that is good. When I compute 20% of the correct answer, I have $0.20 \times 7.75 = 1.55$. So the interval for an acceptable estimate is $7.75 \pm 1.55 = 6.2$ to 9.3 . My estimate for the slope is in that interval, so it is acceptable.