EXERCISE SET 10.1 STUDENT: ______________________
MATD 0370
DUE DATE: ____________________ INSTRUCTOR: ______________________

10.1A Square Roots of Integers

The square roots of a number are the values which, when squared, result in that number.
If $\sqrt{k}$ is a square root of k, then $(\sqrt{k})^2 = k$: $(\sqrt{9})^2 = (3)^2 = 9$, $(\sqrt{5})^2 = 5$, etc.

Every positive number has two square roots, one positive and the other negative. For example, the number 25 has two square roots, 5 and –5, because the square of each of these is 25. The *radical* symbol, $\sqrt{\ }$, is used to indicate the positive (or *principal*) square root of a number: $\sqrt{25} = 5$. The negative square root is represented by taking the opposite of the positive root, e.g., the negative square root of 25 is written $-\sqrt{25} = -5$. The number under the radical symbol, in this case 25, is called the *radicand*.

0 has a single square root ($\sqrt{0} = 0$) and negative numbers have no real number square roots since this would require the square of some real number to be negative. For example, $\sqrt{-25}$ is not a real number.

Perfect squares are numbers such as 0, 1, 4, 9, 16, etc., which are the square of a whole number. That is, $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, etc. Thus, the square roots of perfect squares are integers: $\sqrt{0} = 0$; $\sqrt{1} = 1$; $\sqrt{4} = 2$; $\sqrt{9} = 3$; $\sqrt{16} = 4$; etc.

Not all square roots are equal to integers. The square roots of numbers like 5 (which is not a perfect square) are irrational numbers. The only way to represent them *exactly* is by using a radical, $\sqrt{5}$. For calculation purposes, however, we often use a calculator to get a decimal approximation: $\sqrt{5} = 2.236067977...$ or, rounded to three places, 2.236.

Even without a calculator, we can get a rough approximation of such square roots by making use the fact that if $a < b < c$, then $\sqrt{a} < \sqrt{b} < \sqrt{c}$ (square roots preserve the order.)
Since $4 < 5 < 9$, then $\sqrt{4} < \sqrt{5} < \sqrt{9}$. Thus, $\sqrt{5}$ lies between 2 and 3 (i.e., between $\sqrt{4}$ and $\sqrt{9}$.)

More generally, to approximate $\sqrt{k}$, locate k between successive perfect squares. Then $\sqrt{k}$ must lie between their square roots. An example follows:

> Between which two consecutive integers does $\sqrt{57}$ lie?
> First, find the two successive perfect squares between which 57 lies. In this example, 57 lies between perfect squares 49 and 64: $49 < 57 < 64$
> $$\sqrt{49} < \sqrt{57} < \sqrt{64}$$
> $$7 < \sqrt{57} < 8$$
> Thus, $\sqrt{57}$ lies between 7 and 8.

Many square roots of non-perfect-squares may be rewritten in simpler form. If k is not a perfect square, but k = a•b where a or b is a perfect square other than 0 or 1, then we may simplify $\sqrt{k}$ by making use of the following rule for square roots:

For non-negative real numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$ $(= \sqrt{a} \bullet \sqrt{b})$

Example: Since 28 = 4•7, $\sqrt{28} = \sqrt{4 \bullet 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

Thus, to simplify the square root of a positive number which is not itself a perfect square, we try to find a perfect square (4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, etc.) other than 0 or 1 which divides evenly into the radicand. If we find such a perfect square factor, we rewrite the radicand as a product of two factors (the perfect square and the number we get when we perform that division). Now, use the rule above to rewrite the square root as a product of two square roots. Take the square root of the perfect square, and either write it in front of the other square root or multiply it by any number immediately in front of the roots. Leave the other factor, which is not a perfect square, under its radical. Check to see if any perfect squares other than 0 or 1 divide evenly into the factor remaining under the radical. If so, repeat the process. If not, then you are finished simplifying the square root. Here are some additional examples:

a. $\sqrt{45} = \sqrt{9 \bullet 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$

b. $7\sqrt{75} = 7\sqrt{25 \bullet 3} = 7\sqrt{25}\sqrt{3} = 7 \bullet 5\sqrt{3} = 35\sqrt{3}$

c. $2\sqrt{99} = 2\sqrt{9 \bullet 11} = 2\sqrt{9}\sqrt{11} = 2 \bullet 3\sqrt{11} = 6\sqrt{11}$

d. $\sqrt{72} = \sqrt{9 \bullet 8} = \sqrt{9}\sqrt{8} = 3\sqrt{8} = 3\sqrt{4 \bullet 2} = 3\sqrt{4}\sqrt{2} = 3 \bullet 2\sqrt{2} = 6\sqrt{2}$

NOTE: The reason $\sqrt{72}$ above could be simplified further is that it has a perfect square factor larger than 9. If you choose the largest perfect square factor, then it will not simplify further: $\sqrt{72} = \sqrt{36 \bullet 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$

10.1B Solving Quadratic Equations by Using the Quadratic Formula

The Quadratic Formula is a formula by which any quadratic equation can be solved, including even those which cannot be solved by factoring.

If a quadratic equation is written in standard form, $ax^2 + bx + c = 0$, where $a \neq 0$, then its solution(s) can be obtained by substituting the values of a, b, and c into the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that a, b, c are the coefficients of the x^2, x, and constant terms.

Example 1: Solve using the quadratic formula: $4x^2 = 6 - 5x$

$$4x^2 = 6 - 5x$$
$$4x^2 + 5x - 6 = 0$$

Rewrite in standard form so that we can determine the values of a, b, and c: Move all terms to one side of the equation, written by descending degree, equal to 0.

Then a = 4, b = 5, and c = –6. Substitute these values into the quadratic formula:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-6)}}{2(4)} = \frac{-5 \pm \sqrt{25 + 96}}{8} = \frac{-5 \pm \sqrt{121}}{8} = \frac{-5 \pm 11}{8}$$

This gives two answers: $x = \frac{-5 + 11}{8} = \frac{6}{8} = \frac{3}{4}$, or

$$x = \frac{-5 - 11}{8} = \frac{-16}{8} = -2$$

Note: This equation could also have been solved by factoring:

$$4x^2 + 5x - 6 = 0$$

$(4x - 3)(x + 2) = 0$ — Factor the trinomial.

$4x - 3 = 0 \quad | \quad x + 2 = 0$ — Set each factor = 0.

$x = \frac{3}{4} \quad | \quad x = -2$ — Solve each linear equation.

Example 2: Solve using the quadratic formula: $m^2 - m - 2 = 4m - 5$

$$m^2 - m - 2 = 4m - 5$$
$$m^2 - 5m + 3 = 0$$

Rewrite in standard form: Move all terms to one side of the equation, collect like terms, write by descending degree, and set equal to 0.

Then a = 1, b = –5, and c = 3. Substitute these values into the quadratic formula:

$$m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{25 - 12}}{2} = \frac{5 \pm \sqrt{13}}{2}$$

This gives two answers: $\frac{5 + \sqrt{13}}{2} \approx 4.303$ and $\frac{5 - \sqrt{13}}{2} \approx 0.697$.

Note that $\frac{5 + \sqrt{13}}{2}$ and $\frac{5 - \sqrt{13}}{2}$ are *exact answers*, while 4.303 and 0.697 are *approximations* obtained using a calculator and rounding to three decimal places.

Example 3: Solve using the quadratic formula: $y^2 + 6y - 2 = 5(y^2 - 1)$

To use the quadratic formula, we must first put the given equation in standard form so that we can determine the values of a, b, and c.

$y^2 + 6y - 2 = 5(y^2 - 1)$	
$y^2 + 6y - 2 = 5y^2 - 5$	Distribute the 5.
$0 = 5y^2 - 5 - y^2 - 6y + 2$	Move all terms to one side of the equation and set equal to 0. We moved them to the right so that the *lead coefficient*, 4, would be positive, which is optional.
$0 = 4y^2 - 6y - 3$	Collect like terms and write by descending degree.

Then a = 4, b = –6, and c = –3. Substitute these values into the quadratic formula:

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(-3)}}{2(4)} = \frac{6 \pm \sqrt{36 + 48}}{8} = \frac{6 \pm \sqrt{84}}{8}$$

Now simplify the square root: $\dfrac{6 \pm \sqrt{84}}{8} = \dfrac{6 \pm \sqrt{4 \bullet 21}}{8} = \dfrac{6 \pm 2\sqrt{21}}{8}$

Finally, simplify (reduce) the fraction by factoring numerator (and denominator) and canceling the common 2:

$$y = \frac{6 \pm 2\sqrt{21}}{8} = \frac{2(3 \pm \sqrt{21})}{2(4)} = \frac{3 \pm \sqrt{21}}{4}$$

This gives two answers: $\dfrac{3 + \sqrt{21}}{4} \approx 1.896$ and $\dfrac{3 - \sqrt{21}}{4} \approx -0.396$

Note that we could have also written each term in the numerator separately over the denominator and then simplified:

$$y = \frac{6 \pm 2\sqrt{21}}{8} = \frac{6}{8} \pm \frac{2\sqrt{21}}{8} = \frac{3}{4} \pm \frac{\sqrt{21}}{4}$$

which is another way to write the exact answer. This form of the exact answer gives the same decimal approximations as the other.

Example 4: Solve using the quadratic formula: $x^2 + 6x + 9 = 0$.

Then a = 1, b = 6, and c = 9. Substitute these values into the quadratic formula:

$$x = \frac{-(6)\pm\sqrt{(6)^2 - 4(1)(9)}}{2(1)} = \frac{-6 \pm\sqrt{36 - 36}}{2} = \frac{-6 \pm\sqrt{0}}{2} = \frac{-6}{2} = -3$$

Note that in this case (involving a perfect square trinomial) we get only one answer. Doing the same problem by factoring shows us that there actually are two answers but they are not different. We just get the same answer twice.

$x^2 + 6x + 9 = 0$		
$(x + 3)^2 = 0$		Factor the trinomial.
$x + 3 = 0$	$x + 3 = 0$	Set each factor = 0.
$x = -3$	$x = -3$	Solve each linear equation.

Example 5: Solve using the quadratic formula: $n^2 + n + 1 = 0$.

Then a = 1, b = 1, and c = 1. Substitute these values into the quadratic formula:

$$n = \frac{-(1)\pm\sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm\sqrt{1 - 4}}{2} = \frac{-1 \pm\sqrt{-3}}{2}.$$ No real solution.

This has no real number solution since it involves $\sqrt{-3}$, which is not a real number.

10.1C MORE APPLICATIONS OF QUADRATIC EQUATIONS

The following examples are applications of quadratic equations.

Example 6: Five times a number is 24 less than the square of that number. Find all such numbers.

Let n = the number.
Then $5n = n^2 - 24$.

Write this equation in standard form by subtracting 5n from both sides:
$0 = n^2 - 5n - 24$, or $n^2 - 5n - 24 = 0$.

You may solve this equation either by factoring or by using the quadratic formula.
Factoring gives $(n + 3)(n - 8) = 0$.

Set each factor equal to zero: $n + 3 = 0$, or $n - 8 = 0$.
Then solve each of these equations to get $n = -3$ or $n = 8$.

The number is –3 or 8 (you must give both answers).

Check the answers in the original problem:
$5(-3) = (-3)^2 - 24$ which gives $-15 = -15$ ✓, and
$5(8) = (8)^2 - 24$ which gives $40 = 40$ ✓.

Example 7: The product of two consecutive odd integers is 195. Find the integers.

Let x = the first odd integer.
Then $x + 2$ = the next odd integer (because odd numbers are two apart).

The equation is $x(x + 2) = 195$.

Distribute the x, and then subtract 195 from both sides of the equation to write the equation in standard form: $x^2 + 2x - 195 = 0$.

Again, you may solve this equation either by factoring or by using the quadratic formula. Factoring gives $(x + 15)(x - 13) = 0$.
Set each factor equal to zero: $x + 15 = 0$, or $x - 13 = 0$.
Then solve each of these equations to get $x = -15$ or $x = 13$.

For each of these answers, find the next odd integer by adding two to your answer:
For the first answer, –15, the next odd integer is $x + 2 = -15 + 2 = -13$.
For the second answer, 13, the next odd integer is $x + 2 = 13 + 2 = 15$.

The odd integers are –15 and –13, or the odd integers are 13 and 15 (you must give both pairs of answers).

Check the answers by multiplying:
$(-15)(-13) = 195$ ✓, and
$(13)(15) = 195$ ✓.

Example 8: The length of one leg of a right triangle is 9 meters. The length of the hypotenuse is three meters longer than the other leg. Find the length of the hypotenuse and the length of the other leg.

Let x = the length of the other leg in meters.
Then $x + 3$ = the length of the hypotenuse in meters.

Use the Pythagorean Theorem, $a^2 + b^2 = c^2$, to solve the problem:
$x^2 + 9^2 = (x + 3)^2$.
When simplified, the equation becomes $x^2 + 81 = x^2 + 6x + 9$ (don't forget that the square of a binomial is a trinomial).

When x^2 is subtracted from both sides of the equation, the equation becomes:
$81 = 6x + 9$, which is no longer a quadratic equation because it has no x^2 term.

Solve as a linear equation by subtracting 9 from both sides and then dividing by 6:

$72 = 6x$, or $\frac{72}{6} = \frac{6x}{6}$, which becomes $12 = x$.

Therefore, the length of the other leg is $x = 12$ meters.
The length of the hypotenuse is $x + 3 = 12 + 3 = 15$ meters.

Check by using the Pythagorean Theorem:
$12^2 + 9^2 = 15^2$, or
$144 + 81 = 225$, or
$225 = 225$ ✓.

Between which two consecutive integers do each of the following square roots lie?

1. $\sqrt{11}$
2. $\sqrt{28}$
3. $\sqrt{76}$
4. $\sqrt{137}$

Simplify each of the following square roots.

5. $\sqrt{52}$
6. $\sqrt{45}$
7. $\sqrt{48}$
8. $\sqrt{150}$
9. $5\sqrt{12}$
10. $3\sqrt{24}$
11. $2\sqrt{63}$
12. $7\sqrt{80}$

Solve the quadratic equations below either by factoring or by using the quadratic formula. Give exact answers, and where appropriate, give approximations rounded to three decimal places.

13. $x^2 + 5x = 2$
14. $2y^2 + 4y = 7y + 8$
15. $3m^2 + 1 = 9 - 2m$
16. $5 + 9x = 2x^2$
17. $n^2 - 7n + 5 = 8$
18. $10x^2 - 15x = 0$
19. $5(y^2 - 2y) + 1 = y + 4$
20. $1 - 3m^2 = m^2 - 2(m + 1)$
21. $3u^2 = 6(3u - 1)$
22. $5x - 3(2x - 7) = 4x^2 + 17$

Solve each of the following applications:

23. Four times a number is 12 less than the square of that number. Find all such numbers.

24. The product of two consecutive odd integers is 143. Find the two integers.

25. The sum of the squares of two consecutive integers is nine less than ten times the larger. Find the two integers.

26. The length of a rectangular garden is 3 feet longer than the width. If the area of the garden is 88 square feet, find the dimensions of the garden.

27. A triangle has a base that is 2 cm longer than its height. The area of the triangle is
12

square cm. Find the lengths of the height and the base of the triangle.

28. For an experiment, a ball is projected with an initial velocity of 48 feet/sec. Neglecting air resistance, its height H, in feet, after t seconds is given by the formula
$H = 48t - 16t^2$
How long will it take for the ball to hit the ground? (Hint: $H = 0$ when it hits the ground.)

29. The length of one leg of a right triangle is 8 inches. The length of the hypotenuse is four inches longer than the other leg. Find the length of the hypotenuse and the length of the other leg.

30. A water pipe runs diagonally under a rectangular garden that is one meter longer than it is wide. If the pipe is 5 meters long, find the dimensions of the garden.

ANSWERS:

1. 3 and 4 2. 5 and 6 3. 8 and 9 4. 11 and 12

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5. $2\sqrt{13}$
6. $3\sqrt{5}$
7. $4\sqrt{3}$
8. $5\sqrt{6}$
9. $10\sqrt{3}$
10. $6\sqrt{6}$
11. $6\sqrt{7}$
12. $28\sqrt{5}$

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13. $x = \dfrac{-5+\sqrt{33}}{2} \approx 0.372$, or $x = \dfrac{-5-\sqrt{33}}{2} \approx -5.372$
14. $y = \dfrac{3+\sqrt{73}}{4} \approx 2.886$, or $y = \dfrac{3-\sqrt{73}}{4} \approx -1.386$
15. $m = \dfrac{4}{3}$, or $m = -2$
16. $x = -\dfrac{1}{2}$, or $x = 5$
17. $n = \dfrac{7+\sqrt{61}}{2} \approx 7.405$, or $n = \dfrac{7-\sqrt{61}}{2} \approx -0.405$
18. $x = 0$, or $x = \dfrac{3}{2}$
19. $y = \dfrac{11+\sqrt{181}}{10} \approx 2.445$, or $y = \dfrac{11-\sqrt{181}}{10} \approx -0.245$
20. $m = \dfrac{1+\sqrt{13}}{4} \approx 1.151$, or $m = \dfrac{1-\sqrt{13}}{4} \approx -0.651$
21. $u = 3+\sqrt{7} \approx 5.646$, or $u = 3-\sqrt{7} \approx 0.354$
22. $x = \dfrac{-1+\sqrt{65}}{8} \approx 0.883$, or $x = \dfrac{-1-\sqrt{65}}{8} \approx -1.133$

ANSWERS:

23. The number is –2 or 6. [Equation is $4x = x^2 - 12$]

24. The odd integers are 11 and 13, or the odd integers are –13 and –11. [Equation is $x(x + 2) = 143$]

25. The consecutive integers are 0 and 1, or the consecutive integers are 4 and 5. [Equation is $x^2 + (x + 1)^2 = 10(x + 1) - 9$]

26. The width is 8 feet, and the length is 11 feet. Note that dimensions of geometric figures cannot be negative. [Equation is $x(x + 3) = 88$]

27. The height is 4 cm, and the base is 6 cm. [Equation is $\frac{1}{2}x(x + 2) = 12$]

28. The ball will hit the ground in 3 seconds. [Equation is $0 = 48t - 16t^2$]

29. The other leg is 6 inches, and the hypotenuse is 10 inches. [Equation is $x^2 + 8^2 = (x + 4)^2$]

30. The width is 3 meters, and the length is 4 meters. [Equation is $x^2 + (x + 1)^2 = 5^2$]