Matrix Notation

Notation used to denote elements in a matrix: $a_{ij}$

Two matrices are equal if
i) they have the same dimension
ii) corresponding elements are equal

Examples:

Operations with Matrices:

Addition/Subtraction

Only matrices with the same dimension can be combined. In either case the result is a matrix with the same dimension

Scalar Multiplication

Suppose $k$ is a real number.

Example: Given $B = \begin{bmatrix} 4 & 0 & 6 \\ 1 & -1 & 0 \\ 2 & 9 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 1 & 20 \\ 1 & -3 & 0 \\ 0 & 10 & 3 \end{bmatrix}$,

Find the following:

(a) $B + C$. 
(b) $2B - 3C$. 
Matrix Multiplication

The product of an \( m \times n \) matrix \( A \) and a \( n \times p \) matrix \( B \), is the \( m \times p \) matrix \( AB \). To find the element \( a_{ij} \) in \( AB \), multiply each element in the \( i \)th row of \( A \) by the corresponding element in the \( j \)th column of \( B \). The sum of these products will be \( a_{ij} \).

Example: Given

\[
A = \begin{bmatrix}
0 & 5 \\
4 & 0 \\
-1 & -2 \\
3 & 6
\end{bmatrix}, \quad B = \begin{bmatrix}
4 & 0 & 6 \\
1 & -1 & 0 \\
2 & 9 & -2
\end{bmatrix}, \quad C = \begin{bmatrix}
-2 & 1 & 20 \\
1 & -3 & 0 \\
0 & 10 & 3
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
-4 & 8 & -3 & 7 \\
3 & 0 & -2 & 9
\end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix}
-3 & 0 \\
5 & 2
\end{bmatrix}
\]

Find the following products, if possible.

(a) \( A \cdot E \) \quad (b) \( B \cdot E \) \quad (c) \( B \cdot C \)

Applications

Example: Tuition Costs  p 548 # 67

Example: Car Sales p 548 # 70

Example: A manufacturer produces three models of a product, which are shipped to two warehouses. The number of units \( i \) that are shipped to warehouse \( j \) is represented by \( a_{ij} \) in matrix \( A \) below. The price per unit is represented by matrix \( B \). Find the product \( BA \) and interpret the results.

\[
A = \begin{bmatrix}
1000 & 3000 \\
2000 & 4000 \\
5000 & 8000
\end{bmatrix} \quad B = \begin{bmatrix}
$25 & $20 & $32
\end{bmatrix}
\]
Section 6.6 Inverses of Matrices

For an interesting understanding of matrix inverses, please read the computer graphics application which begins this section.

The Identity Matrix

The \( n \times n \) identity matrix, denoted by \( I_n \)

Matrix Inverses:

Suppose \( A \) is an \( n \times n \) matrix. If the inverse of \( A \), denoted \( A^{-1} \) exists, then it is the \( n \times n \) matrix satisfying \( A \cdot A^{-1} = A^{-1} \cdot A = I_n \)

If \( A^{-1} \) exists, then it is called invertible or nonsingular.
If \( A \) does not have an inverse, then it is singular

Example: Verify that \( A \) and \( B \) are inverses if

\[
A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}
\]

Finding an inverse symbolically

Example: Find \( A^{-1} \) if \( A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \)

Start by forming the 3 x 6 augmented matrix:

Next, use row operations to obtain \( I_3 \) on the left side of the augmented matrix

The right side is the inverse.

Check your work by showing that the right side matrix satisfies \( A \cdot A^{-1} = A^{-1} \cdot A = I_n \)

Example: Find \( A^{-1} \) for the 2 x 2 matrix \( A = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \)
Solving Linear Systems with Inverses and Matrix Equations

A matrix equation can be used to represent a system of linear equations. Consider the following system:

\[
\begin{align*}
2x + y - 3z &= 1 \\
y - 4z &= 6 \\
-3x + y + 5z &= 4
\end{align*}
\]

Let \( A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -4 \\ -3 & 1 & 5 \end{bmatrix} \) be the coefficient matrix.

Let \( X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) be the variable matrix and \( B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \) be the constant matrix.

Find the product \( AX \).

The matrix equation \( AX = B \) is equivalent to the original system. Any linear system can be represented by a matrix equation.

Solving for \( X \) in this matrix equation should give us the solution to the system.

Special note: Since matrix multiplication is not commutative, you the student must remember to multiply each side of the matrix equation on the left by \( A^{-1} \). In general \( A^{-1} \cdot B \neq B \cdot A^{-1} \).

Example: Solve the following system using a matrix equation.

\[
\begin{align*}
x - 3y &= 2 \\
2x + 4y &= -6
\end{align*}
\]
Example: Solve the following system using a matrix equation.

\[
\begin{align*}
  x + y - z &= 0 \\
  2x - y - z &= 2 \\
  -x + y + z &= 4
\end{align*}
\]

Applications

Example: p 561 # 71